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UNIVERSITY OF ILLINOIS BULLETIN

ISSUED WEEKLY

Vol. XXVI

November 20, 1928

No. 12

[Entered as second-class matter December 11, 1912, at the post office at Urbana, Illinois, under the Act of August 24, 1912. Acceptance for mailing at the special rate of postage provided for in section 1103, Act of October 3, 1917, authorized July 31, 1918.]

A STUDY OF THE FAILURE OF CONCRETE UNDER COMBINED COMPRESSIVE STRESSES

BY

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BULLETIN No. 185
ENGINEERING EXPERIMENT STATION

PUBLISHED BY THE UNIVERSITY OF ILLINOIS, URBANA

PRICE: FIFTY-FIVE CENTS

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UNIVERSITY OF ILLINOIS,
URBANA, ILLINOIS

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A STUDY OF THE FAILURE OF CONCRETE UNDER COMBINED COMPRESSIVE STRESSES

I. INTRODUCTION

1. *Introduction.*—This bulletin deals with an investigation of the failure of concrete under compressive stresses applied in one, in two, or in three directions perpendicular to each other, the aim of the tests being to study the internal action of the material as it breaks down under compressive stress, and at the same time to obtain information on the influence of lateral stresses upon the ability of concrete to resist longitudinal stresses. The investigation has been carried out in two parts: (1) a study of concrete specimens loaded in compression in one, in two, or in three directions at right angles to each other by means of fluid pressures, and (2) a study of plain and spirally reinforced concrete compression specimens loaded in one direction in the usual manner. The bulletin contains the results of the first group of tests (Series Nos. 2, 3A, and 3B), together with a statement of various theories of failure applying to the subject, and a critical study of their applicability to the problem. It is planned to report the second group of tests (Series No. 1) in a later bulletin.

2. *Acknowledgments.*—Most of the tests reported in this bulletin were carried on in 1925-26 under the direction of PROF. ARTHUR N. TALBOT, then in charge of the Department of Theoretical and Applied Mechanics. Professor Talbot gave a great deal of attention to the planning and performance of the tests and to the interpretation of the results, and has given valuable assistance and criticism in the preparation of this bulletin. Acknowledgment is also made to PROF. H. F. MOORE who designed the hydraulic pressure apparatus used in the tests.

It seems desirable to make a statement regarding the division of the work among the authors, a distinction rather difficult to make inasmuch as this investigation required to an unusual degree the help of the entire concrete laboratory staff. A large part of the investigation was embodied in a Master's thesis by MR. BRANDTZAEG, who developed a theory of failure of materials which led to the experimental studies described herein. Mr. Brandtzaeg is largely responsible for the outline of the investigation and for much of the analysis of the test results. MR. BROWN devised most of the measuring apparatus used, and

supplied a technique of testing which was invaluable to the success and reliability of the work. The work was carried on under the supervision of PROF. RICHART, who gave personal attention to the development of test methods and performance, as well as to the interpretation of results and to the preparation of this bulletin.

The work was performed as a part of the regular program of the Engineering Experiment Station under the administrative direction of DEAN M. S. KETCHUM, Director of the Station, and of PROFESSORS A. N. TALBOT and M. L. ENGER, successive heads of the Department of Theoretical and Applied Mechanics.

3. Scope of Investigation.—As indicated in the introduction, one of the objects of this study was to ascertain how the ability of concrete to resist stress in one direction is influenced by the presence of stresses in other directions. This question has many direct applications to problems of design, such as in flat slab construction, in arch dams, and particularly in reinforced concrete columns. A partial solution of the problem might be secured by tests of the material under certain definite combinations of stress for which quantitative results could be obtained; however, since there are an infinite number of possible stress combinations, a very great variety of tests would be required to furnish comprehensive information. In the present studies it seemed preferable to use a relatively small number of tests as a means of investigating the validity of a general conception of the process of failure. If the hypothesis is verified, it should then furnish a means of estimating the effect of various stress situations; moreover, a conception of the mechanism of failure should have a very direct bearing on matters such as methods of proportioning concrete for desired strength, durability, and like properties. The part of the investigation included in this bulletin consists of three principal series of tests, as follows:

Series No. 2

This series consisted of tests of forty-eight 4 by 22-in. concrete cylinders subjected to a two-dimensional compression produced by a liquid pressure on the curved sides, with companion tests of 4 by 8-in. concrete cylinders loaded in simple axial compression in a testing machine.

Series No. 3A

This series consisted of tests of sixty-four 4 by 8-in. concrete cylinders in three dimensional compression, two of the principal stresses being equal, and smaller than the third one. The two smaller stresses were applied by liquid pressure on the sides of the cylinder; the larger axial stress was applied to the cylinder in a testing machine.

Series No. 3B

This series consisted of tests of forty-eight 4 by 22-in. and 4 by 8-in. concrete cylinders in three-dimensional compression, two principal stresses being equal and larger than the third one. The two larger lateral stresses were applied by liquid pressure, the smaller axial one in the testing machine.

4. *Theories of Failure of Materials under Combined Stress.*—The circumstances attending the failure of materials under combined stresses have been the subject of extensive investigation, and a number of theories have been proposed by which a criterion of the conditions producing failure might be established. The present investigation deals with a brittle material subjected to combined compressive stresses, and the following discussion applies mainly to this limited field of conditions.

Certain of the theories of failure consist of the assumption of a simple criterion of failure which seems to have some agreement with the results of tests. Examples of theories of this sort are the "Maximum Stress Theory," the "Maximum Strain Theory" and the "Maximum Energy Theory." A second group of failure theories is based on a study of the manner in which failure takes place, and the criteria of failure thus derived are generally less simple in application than are those that are assumed at the outset. In this group of theories an outstanding characteristic is the conception of failure as involving a sliding on planes of least resistance which are inclined to the directions of the principal stresses. The group includes Coulomb's "Internal Friction Theory" (of which Guest's "Maximum Shear Theory" is a special case and Becker's combined "Maximum Strain-Shear Theory" is a further modification, though it may also be properly classed in the first group above), "Mohr's Theory," which is a generalization of Coulomb's hypothesis, and the theories of Böker and Brandtzaeg which are based on the assumption of a non-isotropic material. Used in connection with a brittle material such as concrete some of the foregoing theories are obviously inapplicable.

The maximum stress theory, in which it is assumed that failure occurs when the stress in any direction exceeds a limiting value for the material, was first applied to the problem of combined stresses by Lamé and Clapeyron, and later by Rankine. It is obviously incorrect as regards the condition of three-dimensional compression, and has been disproved for other stress situations as well.

The maximum strain theory involves the assumption that a certain limiting deformation, rather than a limiting stress, will bring

about the failure of a material. This theory, originated by St. Venant, and applied particularly with reference to a maximum tensile strain or elongation, has been used widely as applying to brittle materials. As will be noted in a later chapter, a comparison of the strains measured and the loads carried in one- and two-dimensional tests of concrete show decided disagreement with the values to be expected from the use of this theory.

Various forms of the maximum energy theory have been proposed. The energy of deformation of a unit volume of material has been used to investigate the relations existing between certain elastic properties of the material under simple stress and under combined stress. The theory is applied to elastic failure rather than to the ultimate resistance of the material to stress; for a material like concrete, which shows an appreciable amount of inelastic deformation even at comparatively low loads, the energy theory will have very limited application.

A modification of the energy relation, in which only the energy of deformation used in producing angular deformations is considered, was advanced by Hencky* for use with materials which fail by a "sliding" failure. It was not intended to apply to the case of equal tension in three principal directions, wherein no angular deformations would be produced. The latter case would be included in the energy-criterion advanced by Huber,† who assumed that in the case of tension the sum of the energy used for change of shape, or angular deformation, and that used to produce increase in volume, should be taken as the criterion of failure, while in the case of compression only the energy of angular deformation should be considered.

The internal friction theory, originated by Coulomb, was the first of a group of theories based on the conception of failure as a sliding along planes inclined to the direction of the principal stresses. The resistance to sliding is assumed to consist of two parts: a constant shearing strength and a resistance of the nature of friction that is proportional to the normal stress on the plane of sliding. The material is assumed to fail when the actual shearing stress on any plane exceeds the sum of the shearing strength and the frictional resistance. The slip lines appearing on the surface of mild steel during yielding (generally known as Lüders' lines) and the appearance of fractures of brittle materials on inclined planes or conical surfaces have generally been taken to be a substantiation of the internal friction theory.

*Hencky, H., "Zur Theorie plastischer Deformationen und der hierdurch im Material hervorgerufenen Nebenspannungen," Proc. 1st. International Congress for Applied Mechanics; Delft, Holland, 1924.

†See Föppl, A. and L., "Drang und Zwang," Vol. 1, p. 50, 1924.

The maximum shear theory, proposed by Guest,* is a special case of Coulomb's theory in which the influence of the internal friction is neglected. As shown by Becker† in tests of mild steel, the maximum shear theory apparently applies for certain combinations of stress, and the maximum strain theory for other combinations.

The theory advanced by Mohr‡ is a generalization of Coulomb's theory, and thus includes both the internal friction and the maximum shear theories as special cases. The theory is based (as is also the internal friction theory) on the assumptions that the material acts like a homogeneous material and that there is a relation between the normal stress σ and the shearing stress τ in any plane which governs the resistance to failure along that plane. The limiting values of the normal and shearing stresses are interdependent; thus, the limiting value of the shearing stress, above which failure will occur, must be a function of the simultaneous value of the normal stress on the same plane, or algebraically

$$\tau = f(\sigma) \quad (1)$$

Now considering the plane along which the shearing stress τ is a maximum, it may be shown that with any combination of stresses the maximum shearing stress acts along a plane normal to the plane of the algebraically greatest and least principal stresses, and is therefore independent of the intermediate principal stress. This leads to the important conclusion, pointed out by Mohr, that the failure of a material depends only on the value of the two extreme principal stresses, the (algebraically) greatest and the least; this conclusion, of course, is also implied in the internal friction and maximum shear theories. There is a very evident advantage in the reduction of the three-dimensional problem to that of two principal stresses in a single plane.

Mohr's theory applies primarily to materials that fail by a sliding deformation. It is evident that failures may also occur by a splitting action, such failures being due to tensile stresses. These two typically different actions, a sliding and a splitting failure, may take place within the same material, and it seems reasonable that they may be governed by different laws, such as a "plane of least resistance" law for the sliding or shearing actions and a "maximum strain" or "maximum stress" law for the splitting action.

*Guest, J. J., "The Strength of Ductile Materials under Combined Stress," *Philosophical Magazine*, Vol. L, pp. 69-132, 1900.

†"The Strength and Stiffness of Steel under Biaxial Loading," *Univ. of Ill. Eng. Exp. Sta. Bul.* 85, 1916.

‡Mohr, O., "Welche Umstände bedingen die Elastizitätsgrenze und den Bruch eines Materials," *Zeitschrift des Vereins deutscher Ingenieure*, p. 1524, 1900. Also "Abhandlungen aus dem Gebiete der Technischen Mechanik," 2nd. Ed., pp. 192-235, 1914.

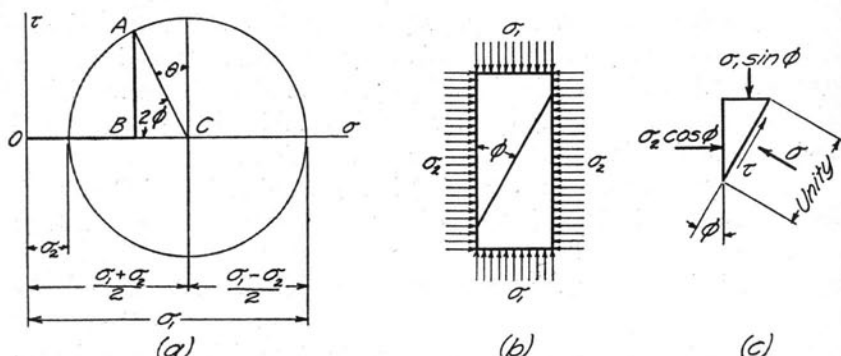


FIG. 1. CONSTRUCTION OF MOHR'S CIRCLE

It will be useful in the comparison of Mohr's theory with test results to use Mohr's method of representing the relation between shearing and normal stress. In Fig. 1 let the ordinates represent shearing stress and the abscissas normal stress. In plotting values of the extreme normal stresses, σ_1 and σ_2 along the σ -axis, draw a circle (generally known as "Mohr's circle") having its center a distance $\frac{\sigma_1 + \sigma_2}{2}$

from the origin and its radius equal to $\frac{\sigma_1 - \sigma_2}{2}$. If any radius be drawn, making an angle 2ϕ with the σ -axis, the point A on the circumference represents the stress combination existing on a plane making an angle ϕ with the principal stress σ_1 . Thus the ordinate AB represents the shearing stress and the abscissa OB the normal stress in the given direction. This is seen by reference to Fig. 1 (b) and (c), which show respectively the principal stresses acting on a body and the components of these stresses along a plane inclined at an angle ϕ with the direction of σ_1 . Considering the triangular element of Fig. 1 (c), it is found that

$$\sigma = \sigma_1 \sin^2 \phi + \sigma_2 \cos^2 \phi = \frac{\sigma_1 + \sigma_2}{2} - \frac{\sigma_1 - \sigma_2}{2} \cos 2\phi \quad (2)$$

$$\tau = \frac{\sigma_1 - \sigma_2}{2} \sin 2\phi \quad (3)$$

By referring to Fig. 1 (a), it is evident that by construction AB is equal to τ and OB is equal to σ . Hence the circle of Fig. 1 (a) may be taken to be the locus of points representing the corresponding values of normal and shearing stress in any possible direction due to the given

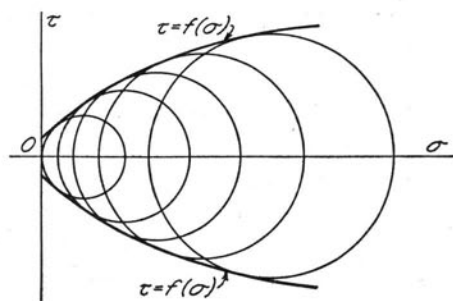


FIG. 2. RELATION OF LIMITING CURVE TO MOHR'S CIRCLES

principal stresses σ_1 and σ_2 . For other values of σ_1 and σ_2 similar circles could be drawn, as shown in Fig. 2. If these values of σ_1 and σ_2 represent the principal stresses at failure, according to Mohr's theory some point on each of these circles must also be on the curve represented by equation (1), and it is evident that the curve must be tangent to all of the circles. There can be no point on the circles falling above the curve, since the material would have failed at an earlier stage, and there must be at least one point on each circle coinciding with the curve, since the material has actually failed under the given principal stresses and, presumably, the dangerous stress-combination on some plane within the material. Hence it is logical to use the curve that is tangent to the stress circles as a limiting curve.

Mohr's theory did not establish the shape of the limiting stress curve. It is seen that if the maximum shear theory were correct, the circles would all have the same diameter, and the limiting curve would be horizontal. If the internal friction theory were applicable, the curve would be an inclined straight line with a τ -intercept equal to the shearing strength of the material and a slope equal to $\cot 2\phi$, or $\tan \theta$, where θ is considered to be the "angle of internal friction." It is evident that the limiting curve must be determined experimentally for any given material.

5. *Investigations of Failure Phenomena by Kármán and Böker.*—The analyses and tests by Kármán* and Böker† at the University of Göttingen probably constitute the most comprehensive study of the theories of failure of a brittle material on record. The tests were made principally to try out the correctness of Mohr's theory, and included

*v. Kármán, Th., "Festigkeitsversuche unter allseitigem Druck," Mitteilungen über Forschungsarbeiten auf dem Gebiete des Ingenieurwesens, Vol. 118, 1912.

†Böker, Robert, "Die Mechanik der bleibenden Formänderung in kristallinisch aufgebauten Körpern." Mitteilungen über Forschungsarbeiten auf dem Gebiete des Ingenieurwesens, Vol. 175 and 176, 1915.

tests of materials under three combinations of stress: (1) cylinders subjected to a large end pressure and smaller lateral pressures, (2) cylinders subjected to a small end pressure and larger lateral pressures, and (3) cylinders subjected to torsion in addition to the lateral and axial pressures. The material used in the tests was marble of a very uniform quality, though in some of the tests sandstone and zinc were also used.

Each of the three groups of tests gave results that were in accordance with Mohr's theory inasmuch as they determined a limiting curve, such as that shown in Fig. 2. However, the limiting curves were different for the three groups of tests; the limiting value of the shearing stress was greater for a given value of the normal stress when the intermediate principal stress was high, as in the second group, than when the intermediate principal stress was low, as in the first group. For the third group, in which an applied torsion produced tensile stress in the specimens, failure was evidently governed by two laws. Up to a certain constant value of the resultant tensile stress Mohr's law apparently governed the failure, and when this limiting resultant tensile stress was reached failure occurred regardless of the other stresses. Thus two conclusions may be derived from these tests: (1) that the failure of a material may be controlled by a dual law, a tensile or splitting failure occurring with certain stress combinations, and a shearing or sliding failure occurring with other combinations; (2) that, contrary to the implications of Mohr's law, failure was not entirely independent of the magnitude of the intermediate principal stress. Böker considered that this discrepancy was due to the lack of isotropy of the material.

The shape of the limiting curves found in the tests is of interest. In the first group of tests it was found that as the normal stress became very large, the curve became nearly straight, and parallel to the σ -axis, indicating agreement with the maximum shear theory. At these pressures (40 000 to 50 000 lb. per sq. in.) a marked change in the crystalline structure of the marble was noted. It appeared that the sliding failure at the lower loads was dependent upon the normal stress and was largely *between* crystals of the material, while at very high pressures it depended upon shearing stress only and was largely failure *within* the crystals.

Böker made an analysis of the possible manner of failure of a material composed of non-isotropic crystalline particles. The material tested, marble, is made up of crystals having their axes oriented arbitrarily. Obviously there is a possibility of failure between crystals or

within them, since each crystal has planes of weakness in some particular direction. Böker considered that there were four possible types of failure, a splitting failure or a sliding failure, between crystals or within crystals; but he placed the greater emphasis on the sliding failure. Since the planes of weakness are inclined at many directions at different points in the material, it follows that a condition may develop under stress in which plastic or inelastic sliding has started at some points, but not at adjacent points. Böker developed analytical relations between the stresses applied and the development of plastic sliding throughout the material, and showed that with the multiplicity of directions of sliding, the intermediate principal stress must have some effect. Obviously, this type of sliding failure does not produce sliding along a single set of parallel plane or conical surfaces throughout the material, as assumed in the various foregoing theories.

6. *A Theory of Failure of a Material Composed of Non-Isotropic Elements.*—Using Böker's analysis of the action of marble as a basis, Brandtzaeg* developed a theory of failure of an ideal material made up of non-isotropic elements. He assumed that these elements may yield plastically through a lamellar sliding action fixed in direction for each element, but varying arbitrarily throughout the material. This sliding occurs when a definite value of shearing stress along the plane of sliding is reached. While this analysis was made of the action of a hypothetical material, it was believed that it would lead to a conception of the action to be expected from a material like concrete, for which the conditions of stress and deformation of elementary parts are undoubtedly extremely irregular, and practically impossible of analysis. It is not intended to imply, however, that the plastic or inelastic deformation of concrete is due to sliding failure within the grains of aggregate.

The hypothetical material is assumed to be originally a continuous, perfectly elastic mass, made up of small elements each of which has planes of weakness or susceptibility to sliding failure in a particular direction. These "directions of weakness" are assumed to vary arbitrarily from one element to another; considering a unit volume of the material, it is assumed that all geometrically possible directions of weakness are represented to an equal degree within that volume. For the purpose of mathematical analysis it may be considered that the plane of weakness of each element is represented by a small element of the surface of a hemisphere, and that the entire surface of the

*Brandtzaeg, Anton, "Failure of a Material Composed of Non-Isotropic Elements," det kgl. Norske Videnskabers Selskabs Skrifter, 1927. Nr. 2, Trondhjem, Norway. The analysis was made in connection with the tests described in this bulletin.

hemisphere thus represents in area and direction the planes of weakness in a unit volume of the material.

The material initially has a modulus of elasticity E and Poisson's ratio μ and deforms elastically under low loads. Until tensile stresses of a certain magnitude are developed the only way for any element to yield, or cease to deform elastically, is by a plastic sliding, or inelastic shearing detrusion, in the one fixed direction of weakness for that element. The law governing the beginning of sliding failure might be characterized as an "internal friction theory" for each individual element; that is, the resistance to sliding τ is taken to be made up of two parts, one term, τ_0 , representing the shearing strength of the material, and the second, $f\sigma$, a constant times the normal stress, as given by Equation (4)

$$\tau = \tau_0 + f\sigma \quad (4)$$

When the shearing stress along the plane of weakness of an element of the material reaches the value τ , plastic sliding of the material follows.

In addition to the possibility of a plastic sliding failure, it is assumed that "splitting" failure may occur whenever the tensile stress in any direction reaches a limiting value σ_t . It is assumed that the material fails abruptly, without any plastic deformation, across a plane normal to the tensile stress. The "planes of weakness" of the elements have been assumed to have no effect upon the resistance of the material to splitting. The splitting type of failure was observed in the tests by Kármán and Böker. While the present analysis has assumed a maximum tensile stress as a criterion of failure, as suggested by the tests by Kármán and Böker, it is quite possible that the maximum tensile strain is the criterion that should have been used.

Reference has been made to "elements" and to "unit volumes" of the material. It is assumed that a unit volume will contain a sufficiently large number of elements to make it possible to apply statistical averages to the properties of the elements. The stress in an element may be considered as an intensity, or individual state of stress, varying, after plastic action begins, from element to element. The stress in a unit volume refers to an average unit stress, as derived from the externally applied load and the cross-section of the unit. Thus the intensity of stress may vary considerably above or below the average unit stress, according to the kind of deformation of a particular element.

With the beginning of application of simple compression to the assumed material, the intensity of stress is uniform throughout all elements and all deform elastically. With increasing load, a few ele-

ments having directions of weakness most favorable to the development of a sliding failure undergo a plastic sliding deformation; the few elements so deforming are arbitrarily scattered throughout an otherwise elastic material and their deformation is governed by that of the adjoining elements, though they do not contribute fully to the load-carrying capacity of the material. It is easily demonstrated that the angle ϕ_o between the direction of weakness of those elements which first fail by sliding and the direction of the greatest principal stress

is $\phi_o = 45^\circ - \frac{\theta}{2}$ where θ is the angle of internal friction for the ele-

ment. As the load and the consequent shearing stress increase, other slightly less favorably situated elements undergo plastic deformation, though as long as the material remains continuous all elements must deform equally.

Although the elastic elements carry more than their share of the axial stress, it is evident that the axial stress in an element increases with the load even after it has begun to deform plastically; this is possible only if a lateral restraint is supplied by the surrounding elastic elements. Conversely, the lateral compression developed in an element in the plastic stage must be balanced by lateral tension in the surrounding elements; this may be characterized as a "wedging" action against which the material is kept from a splitting failure only by the action of elastic elements which form lateral "ties." As plasticity spreads throughout the material with increasing loads, it is evident that the critical lateral tensile stress or strain will be reached at which splitting failure of certain elements will occur. These many small fractures may be expected to combine into continuous cracks running parallel to the direction of the principal compressive stress.

The splitting failure of elastic elements will evidently disturb the equilibrium of those plastic elements that depended upon lateral support to maintain their axial load-bearing ability, and of those the elements in which sliding first began will presumably be most affected; in these elements the loss of lateral support will result in very large plastic deformations. The material at this point may be said to be "disorganized" and in the stage of rapid breaking down. Since sliding failure may be assumed to be most prominent in those elements for which the inclination of the planes of weakness is equal to ϕ_o , it is likely that a general sliding may occur in this direction, the resistance of elements having other directions of weakness being overcome in the process. This may bring about the appearance at complete failure

that has led to the idea of failure along a continuous plane, as assumed in the internal friction theory.

From the foregoing reasoning it is evident that failure in simple compression is due to the simultaneous "splitting" and "disorganizing" effects, the former being of primary importance. In three-dimensional compression, the presence of an external lateral restraint obviously replaces much of the lateral tension in elastic elements and the effect of failure of these elements is less disturbing to the equilibrium of the plastic elements. However, although relatively large plastic deformations may be developed, it appears that failure will eventually be precipitated by the splitting of elastic elements.

A mathematical derivation of general relations as to the stresses and deformations described in the foregoing discussion, for conditions up to the point at which splitting failure begins, is given in the Appendix. Beyond this point the mathematical analysis does not apply; however, the agreement of test results with the analysis within its intended range of application, as discussed in Section 42, forms a very convincing verification of the correctness of the hypothesis advanced regarding the mechanism of failure, as well as of its applicability to a material such as concrete.

7. *Analytical Relations.*—Reference will frequently be made to relations existing between stresses and deformations in an elastic material, since these relations may be applied roughly to the behavior of concrete, at least at low loads. It is particularly useful to note the relative magnitude of the deformations under conditions of biaxial and triaxial loading.

For an elastic body, having a modulus of elasticity E and Poisson's ratio μ when subjected to three principal stresses f_x , f_y , and f_z , it is known from the theory of elasticity that the linear unit deformations ϵ_x , ϵ_y , and ϵ_z corresponding to the directions of the principal stresses are

$$\epsilon_x = \frac{1}{E} [f_x - \mu (f_y + f_z)] \quad (5a)$$

$$\epsilon_y = \frac{1}{E} [f_y - \mu (f_x + f_z)] \quad (5b)$$

$$\epsilon_z = \frac{1}{E} [f_z - \mu (f_x + f_y)] \quad (5c)$$

Referring to the cylindrical specimens tested in this investigation, and using the subscript 1 to denote the axial direction and the subscript 2 to denote the lateral directions,

$$f_x = f_1, f_y = f_z = f_2, \epsilon_x = \epsilon_1 \text{ and } \epsilon_y = \epsilon_z = \epsilon_2,$$

whence

$$\epsilon_1 = \frac{1}{E} (f_1 - 2\mu f_2) \quad (6a)$$

$$\epsilon_2 = \frac{1}{E} [(1 - \mu) f_2 - \mu f_1] \quad (6b)$$

These equations are all linear, so that the presence of a constant stress in one direction does not change the rate of deformation with increase in stress in the other directions. Hence, in Series 3A or 3B, if the material had been elastic, the rate of deformation would have been independent of the magnitude of the lateral principal stress, which was held constant throughout each test.

From Equation (6a), if $f_1 = 0$, as in the case of Series 2, the axial elongation of the specimen is

$$\epsilon_1 = \frac{-2\mu}{E} f_2 \quad (7)$$

and the slope of the pressure—elongation curve (assuming elastic action) in either Series 2 or Series 3B, wherein f_1 is constant, becomes

$$\frac{f_2}{\epsilon_1} = -\frac{E}{2\mu} \quad (8)$$

For the case in which f_1 is equal to f_2 , as in Series 3A during the application of the oil pressure, the unit deformation in all directions is

$$\epsilon_1 = \epsilon_2 = \frac{1 - 2\mu}{E} f_1 = \frac{f_1}{K} \quad (9)$$

wherein K is defined as the "bulk" modulus of elasticity.

The volumetric deformation, or unit change in volume due to the applied stresses, is frequently mentioned; from Equations (6a) and (6b) it is found that the volumetric deformation is

$$\epsilon_v = \epsilon_1 + 2\epsilon_2 = \frac{1 - 2\mu}{E} (f_1 + 2f_2) = \frac{f_1 + 2f_2}{K} \quad (10)$$

The foregoing relations between stresses and deformations in an elastic material will evidently fail of general application to a material such as concrete, in which plastic action begins at low loads; however, the relations should apply to the initial properties of the material, and may be used as an index of the degree of plastic deformation at other stages of loading. While the same notation will also

TABLE 1
TENSILE STRENGTHS OF BRIQUETS
In pounds per square inch

Per cent mixing water used—by weight of dry materials, 25.0 for neat cement, 10.7 for Ottawa sand mortar and 12.0 for Attica sand mortar

No.	Neat Cement		Ottawa Sand Mortar 1:3, by weight		Attica Sand Mortar 1:3, by weight	
	Age, 7 days	Age, 28 days	Age, 7 days	Age, 28 days	Age, 7 days	Age, 28 days
1	702	660	226	330	300	503
2	617	600	203	410	360	455
3	552	740	220	320	355	470
4	605	838	231	300	376	435
5	635	863	225	327	370	435
Average	623	740	221	337	352	460

be used for measured stresses and deformations without implying perfect elasticity of the material, the values of E and μ used throughout the bulletin will refer to the initial values of these quantities, which, determined from the initial slopes of load—deformation curves, should represent definite properties of the material.

II. MATERIALS AND TESTING METHODS

8. *Materials.*—The materials used in the specimens tested were of the same grade as has been used regularly in the laboratory in investigations of concrete and reinforced concrete.

Universal Portland Cement, furnished to the University by the manufacturers, was used in all of the tests. When used it had been stored in a dry room in the laboratory for some time. Immediately before the tests were made a large supply of the cement was taken from the storage room, and screened into tight metal containers, care being taken to secure thorough mixing of all bags of cement. Physical tests of the cement were made according to the Standard Specifications for Portland Cement of the American Society for Testing Materials. The cement passed the soundness test satisfactorily, had a specific gravity of 3.10, a time of initial set of 1 hr. 45 min. and of final set of 6 hrs. 30 min., as determined by the Gillmore needle apparatus, and had 78 per cent passing the No. 200 sieve in the fineness test. The tensile strengths of briquets made with the cement are given in Table 1.

TABLE 2
SIEVE ANALYSES OF AGGREGATES

Material	Sieve Number	Width of Square Sieve Openings in.	Per cent Passing Given Sieve (Data from five samples)		
			Maximum	Minimum	Average
Sand.....	100	0.0058	0	1	1
	48	0.0116	2	8	2
	28	0.0232	19	24	21
	14	0.046	50	58	54
	8	0.093	69	73	71
	4	0.185	92	94	93
Fine Gravel.....	4	0.185	10	13	12
	3	0.263	26	47	40
	$\frac{3}{8}$ -in.	0.371	83	97	93
Coarse Gravel.....	4	0.185	0	1	1
	3	0.263	0	1	1
	$\frac{3}{8}$ -in.	0.371	1	4	3
	$\frac{1}{2}$ -in.	0.525	42	60	48
	$\frac{3}{4}$ -in.	0.742	90	97	94

TABLE 3
PHYSICAL PROPERTIES OF AGGREGATES

Property	Sand	Fine Gravel	Coarse Gravel	Mixed Gravel 1 part fine: 2 parts coarse
Moisture content, as used, per cent..	0.2	0.3	0.3	0.3
Specific gravity.....	2.67	2.69	2.69	2.69
Unit weight, lb. per cu. ft.				
(a). Measured loose, struck off. . .	107	92	90	94
(b). Tamped, A. S. T. M. method*	113	100	100	103
Moisture absorption, per cent.	0.6	1.0	1.0	1.0
Fineness modulus.....	3.55	5.95	7.03	6.67

*Standard Method of Test for Unit Weight of Aggregate for Concrete, A. S. T. M. Book of Standards, Part II., p. 120, 1927.

The sand used was a glacial drift material obtained from near the Wabash River at Attica, Indiana. It was composed of a mixture of calcareous, siliceous, and granitic particles, in the form of hard, smooth and well-rounded grains. As used it was quite dry. Sieve analyses and other physical properties of the sand are given in Tables 2 and 3.

The gravel used was obtained from the same sources as the sand, and was of similar quality and characteristics. It was screened into two parts, one passing a $\frac{3}{8}$ -in. sieve, the other passing a $\frac{3}{4}$ -in. sieve and retained on a $\frac{3}{8}$ -in. sieve. In all cases, two parts of the coarse

gravel were used with one part of the fine gravel. Sieve analyses and other physical properties of the gravel are given in Tables 2 and 3.

9. *Making and Curing of Specimens.*—A total of 8 mixtures of mortar and concrete were used, the proportions of the materials being 1:1, 1:2, 1:3, 1:1:2, 1:2.1:2.5, 1:2:4, 1:3:5, and 1:4:7, by loose volume. Only three of these mixtures were used in Series 3A and 3B. Data regarding the mixtures are given in Table 4, Section 16. All solid ingredients were weighed, and the mixing water was measured by means of a large burette. The mixing was done by hand. The concrete was poured into cylindrical forms made of steel tubing placed on machined cast-iron plates, and tamped according to standard methods now in use in making 6 by 12-in. cylinders.* The specimens for axial compression tests were capped with neat cement when made.

The forms were removed 24 hours after the specimens were poured and the latter were placed in a moist closet for curing. All specimens were tested at the age of 28 days and were usually allowed to dry out in the air of the laboratory for one day before they were tested.

10. *Method of Applying Combined Stresses.*—In choosing the type of specimen to be tested in one-, two-, or three-dimensional compression, a considerable study was made. The essential feature to be secured in such specimens is not merely the establishment of combined stresses, but the certainty that a known state of uniform stress in the different directions may be obtained. This is particularly difficult in the case of a material like concrete. Two general methods that have been used for establishing a uniform state of combined stresses in tests of materials were considered: (1) a method of testing in which the material is compressed between two or three sets of parallel bearing planes, the specimens being cubes or cross-shaped specimens; (2) a method of testing in which one or more of the stresses is applied by means of liquid pressure.

The first method of testing has the disadvantage that the friction upon the bearing surfaces has a very decided effect on the distribution of stress in the specimen. The method has been used with some success in studying the deformations under biaxial compression; it does not seem at all suitable, however, for the case of the three-dimensional stresses. The liquid pressure method of loading which was selected has the disadvantage that the specimen is confined inside

*"Standard Methods of Making Compression Tests of Concrete," A. S. T. M. Book of Standards, Part II, p. 112. 1927.

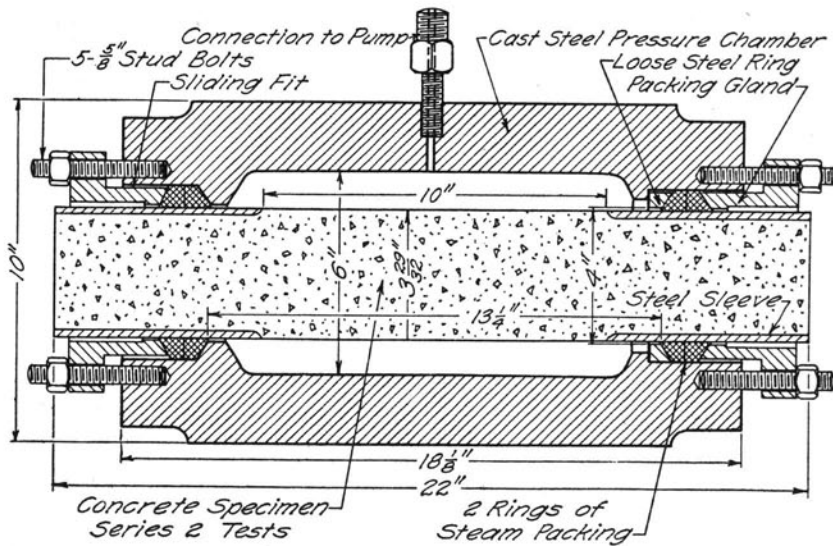


FIG. 3. HYDRAULIC PRESSURE CHAMBER, AS USED IN TESTS OF SERIES 2

a pressure chamber, and can not be seen or measured directly during the test. The method has the great advantage, however, of producing a uniform distribution of stress over the loaded surface, and the influence of friction is not only minimized, but may be roughly determined.

11. *Testing Apparatus.*—The method of producing combined compression used in the tests was that of applying a liquid pressure to the sides of cylindrical concrete specimens, the liquid pressure acting either alone, or combined with an axial load applied by a testing machine.

The liquid pressure was applied to the specimen in a hollow cylindrical chamber of cast steel, of which a longitudinal section is shown in Fig. 3, with a specimen as used in Series 2 in position for testing. A stuffing box at each end of the chamber provided a tight seal between chamber and specimen. The packing of the stuffing boxes was held tightly in place by steel packing glands. At one end there was a clearance of about 0.05 in. between gland and cylinder; at the other end a removable steel ring formed the seat of the stuffing box so that a large opening was provided for the insertion of the test cylinder. The packing used was a fairly pliable graphitic steam packing, $\frac{1}{2}$ -in. square in cross-section. Two rings of this packing

were used in each stuffing box, and were used without renewal for from 5 to 10 tests.

The pressure chamber was designed for an internal working pressure of 7000 lb. per sq. in. The highest pressure commonly used was 4000 lb. per sq. in., since higher pressures gave trouble in maintaining tight joints. Pressure was applied by means of a Watson-Stillman hand pump, a heavy cylinder oil being used as the fluid. The oil pressure was measured at the pump by means of a pressure gage. For the lower pressures, a gage of 3000 lb. per sq. in. capacity, made by the Crosby Steam Gage and Valve Company was used, and for the higher pressures a Watson-Stillman gage of 7000 lb. per sq. in. capacity was used. These gages were calibrated by means of a Crosby fluid pressure scale, which was used as a standard of reference. From a comparison of successive calibrations, it appears that at the pressures used the Crosby gage was accurate to within 20 lb. per sq. in., and the Watson-Stillman gage to within 35 lb. per sq. in.

The pressure chamber represents a type of apparatus that had not been previously tried out in this laboratory. It has certain disadvantages, as seen from the facts that a small axial stress was produced by the friction between the specimen and the packing, a lateral pressure was exerted upon the specimen within the rings of packing, a tight covering on the specimen was required to prevent penetration of oil, and deformation measurements except those in an axial direction were necessarily of an indirect type. In addition, the magnitude of the oil pressures available was rather limited and restricted the range of the tests of Series 3B. In spite of these disadvantages it is felt that the liquid pressure method gave satisfactory results, and is probably the only feasible way of applying three-dimensional compression over a varied range of pressures.

III. CONCRETE IN BIAXIAL COMPRESSION, SERIES 2

12. *Specific Purpose and Outline of Tests.*—In this series of tests concrete and mortar cylinders were loaded by means of uniform liquid pressure applied to their sides with no stress existing in the third direction, a state of purely two-dimensional compression. The series formed an important part of the whole investigation, not only because of many direct practical applications to the problems of design, but also because the relation of the strength of concrete in two-dimensional compression to that in simple compression gives a crucial

test of some of the current theories of failure, particularly the maximum strain theory.

Little experimental information seems to be available regarding the resistance of material under two-dimensional compression. Two tests may be mentioned, the tests by Föppl,* on cubes of mortar and of stone, and tests made by Millard† at the University of Illinois on cubes of concrete. Both of these tests were made by compressing cubes between two sets of greased bearing plates. They agree in general in showing that the strength of concrete in two-dimensional compression is not materially different from the strength in simple compression. However, as previously noted, the friction between specimen and bearing plates raises a serious question as to the validity of tests made in this way.

In Series 2, six 4 by 22-in. specimens, made from each of eight different mixtures, were tested in two-dimensional compression. For each specimen tested in this way, a 4 by 8-in. companion specimen was tested in simple compression for comparison. The 4 by 22-in. specimens were of two types; about half were plain concrete cylinders, and the other half were covered at each end by steel sleeves 6-in. long, made from $3\frac{1}{2}$ -in. steel pipe and turned to an outside diameter of $3\frac{29}{32}$ -in., as shown in Fig. 3. The two types of specimens were used because of some doubt before the tests as to whether the specimen might fail near the area of contact with the packing, and also whether the surface of the concrete would be sufficiently smooth to insure a tight joint and to give the desired low frictional resistance. By using both types of specimens it was hoped to secure information on these factors. At each end of the specimen a short $\frac{5}{8}$ -inch steel rod was inserted to permit the attachment of extensometers.

The principal variable in Series 2 was the concrete mixture. Eight mixtures were used; three mortars of proportions 1:1, 1:2, and 1:3, and five concretes of proportions 1:1:2, 1:2.1:2.5, 1:2:4, 1:3:5, and 1:4:7, by loose volume measurement. All mixtures were of a fairly wet consistency.

13. *Preparation of Test Specimens.*—It was necessary in these tests to apply a surface covering to the specimens to prevent the penetration of oil, since the presence of hydraulic pressure in the pores of the material would induce an axial tensile stress and in many cases would produce a tensile failure. The methods of oil-proofing which

*Föppl, A. Mitteilungen aus dem technischen mechanischen laboratorium, Vol. 27, Munich, 1900.

†Millard, F. H. Master's thesis, University of Illinois, 1912.

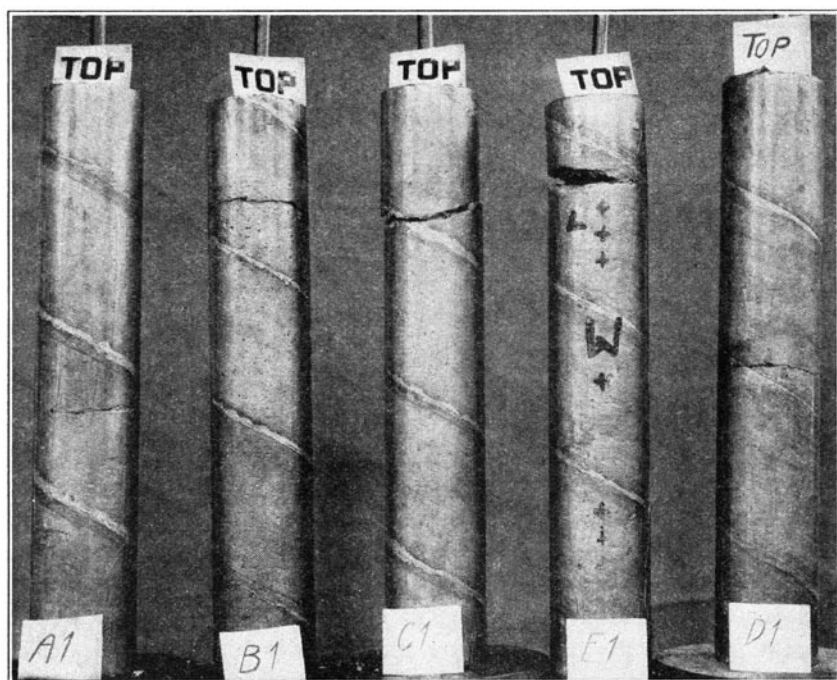


FIG. 4. SPECIMENS OF SERIES 2 WITH BRASS COVERING, AFTER TEST

were tried out in preliminary tests consisted in filling all surface cavities of the specimen with plaster of Paris, and coating the surface with heavy layers of shellac, or with two layers of a high grade rubber cement. These methods were unsatisfactory, probably because the surface coating was easily damaged in inserting the specimen in the pressure chamber. The method finally adopted was that used by Kármán in the tests described in Section 5. The surface of the specimen was covered with a thin envelope of annealed brass. This brass was 0.002 in. thick, and came in strips 6 in. wide, which were applied in spiral form as seen in Fig. 4. The brass was wrapped tightly around the specimen, the joints were tinned and soldered, and then filed down to a smooth surface. In preparing the surface of the concrete some variations in method were used; at first all cavities were filled with plaster of Paris, later it was found desirable to fill the cavities with a rather dry 1:1 Lumnite cement mortar made with sand passing a 100-mesh sieve. This was done several days before testing. After two days of curing, the surface was again rubbed down

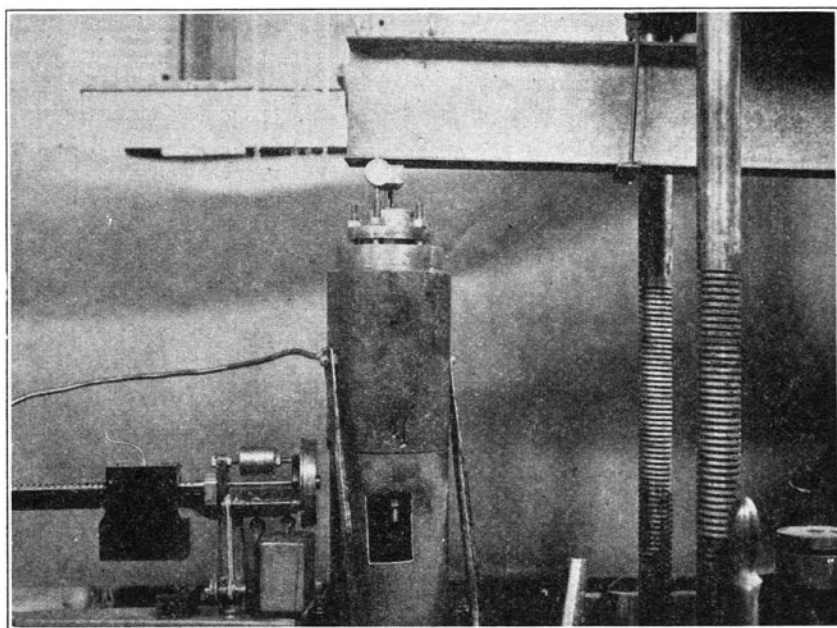


FIG. 5. ARRANGEMENT OF APPARATUS FOR FRICTION TEST, SERIES 2

with a paste of plaster of Paris which removed all visible irregularities. In specimens with steel sleeves, a special treatment was frequently rendered necessary by a settlement crack near the edge of the sleeve. Such cracks were thoroughly cleaned and filled with Lumnite cement and plaster of Paris.

Since measurements of radial deformation could not be taken during the test, observations of permanent deformation were taken by measuring diameters of specimens at fixed gage points before and after the test. Points at which diameters were measured are indicated by crosses in the view of specimen E1 in Fig. 4. The diameters were measured by means of a screw micrometer reading to 0.001 in. and having flat contact surfaces.

14. *Friction Tests and Simple Compression Tests.*—To study the effect of friction between the brass-covered specimen and the packing ring, a series of auxiliary tests was made, using the arrangement of the apparatus shown in Fig. 5. The pressure cylinder was placed vertically on a 10 000-lb. Olsen weighing scale resting on the table of the 300 000-lb. testing machine used to apply the load. The pressure chamber was supported upon a short section of 10-in. pipe,

thus allowing the concrete specimen to slide freely in a vertical direction. At the upper end of the specimen an Ames dial was attached so as to indicate the movement of the specimen relative to the pressure chamber. When load was applied to the specimen, the force exerted was weighed on the 10 000-lb. weighing scale, and the amount of movement was indicated by the Ames dial.

In making a friction test the oil pressure was raised to the desired amount and held constant throughout the test. The specimen was then pushed down through the chamber at a nominal speed of about 0.12 in. per min., and simultaneous readings were taken of the force exerted and of the movement of the specimen. The test was generally carried on until a movement of 0.02 to 0.05 in. had taken place.

Friction tests were made on a total of ten specimens, of which seven were the regular 4 by 22-in. cylinders of the series, and three were extra specimens made of 1:1 Lumnite cement mortar which were tested at the higher oil pressures. All specimens were covered with sheet brass as in the compression test. The result of the friction tests will be described in a later section.

Tests of the 4 by 8-in. simple compression specimens were made in a Riehle 50 000-lb. testing machine. Measurements of deformations were made by the use of an ordinary 5-point extensometer, having a multiplication ratio of 2. The test procedure was the same as that commonly employed with the standard 6 by 12-in. compression cylinder.

15. Biaxial Compression Tests.—In making the fluid pressure test, the steel pressure chamber was laid with its axis horizontal, and the specimen was slid into the chamber on a sheet of heavy paper which was then withdrawn, leaving the specimen in place. The packing and glands were inserted and care was taken in tightening the glands that no bending was produced in the specimen. Extensometers for measuring the axial deformation of the specimen were attached as indicated in Fig. 6. One extensometer was of the strain gage type, in which the lever arm had a multiplication ratio of 4.85, and actuated a 0.001-in. Ames dial. The other was simply an extension of a "Last Word" dial, reading directly to 0.0001 in. Both extensometers were so attached that contacts were held with a fixed pressure against gage holes.

In calculating unit deformations from the extensometer measurements a gage length of 15 in., or roughly the distance between centers of packing rings, was used for the specimens without sleeves; while a

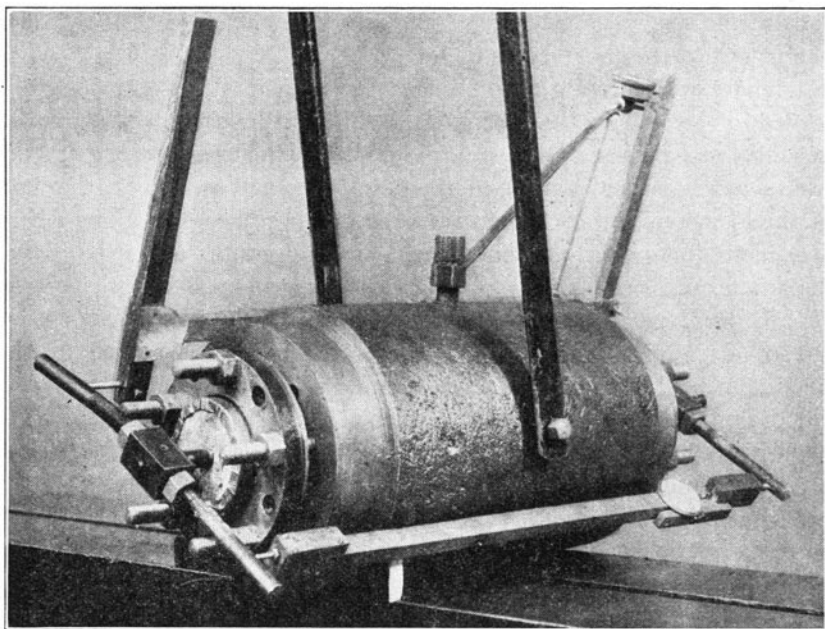


FIG. 6. SPECIMEN IN PRESSURE CHAMBER WITH EXTENSOMETERS ATTACHED, SERIES 2

length of 10 in., or the distance between sleeves, was used for those specimens having sleeves. In a test the oil pressure was applied in increments varying with the strength of the specimen, and the tests were continued until the specimen failed. In removing the latter from the pressure chamber care was taken not to injure the brass covering. Final measurements of diameter were made at the gage lines previously mentioned.

During the early stages of the fluid pressure tests the elongations measured on the extensometers were approximately proportional to the pressure applied; as the pressure increased the elongations generally increased more rapidly. Finally a point was reached where, with the pressure kept constant during the taking of a set of readings, the extensometers failed to come to rest. The pressure at which this yielding or creeping started was recorded. When the creeping started, or soon after, a marked change occurred in the rate of elongation with increase in pressure.

In most cases, failure occurred suddenly as a total fracture across a plane approximately normal to the axis of the cylinder. As soon as a fracture started, the unbalanced oil pressure acting on the broken

ends of the specimen ordinarily forced the two portions of the cylinder partly out of the pressure chamber.

There were certain interesting differences in the phenomena observed in the tests of the specimens of different mixtures. The amount of additional pressure that could be applied after the creeping started was a much larger portion of the total pressure carried in the case of the leaner mixtures than in the case of the richer ones. The richer specimens frequently showed a slight creeping at three-fourths of the maximum pressure, and the amount of creeping did not increase rapidly until very near the maximum load. The leaner mixtures, on the other hand, frequently showed a creeping at one-third of the maximum load. This creeping increased quite rapidly and the rate of elongation became very high. In some cases, the specimens of the leaner mixtures were deformed or squeezed down so much that the pressure could not be held at the packing joints. After removing the brass covering, it was found that these specimens had been cracked near the section encircled by the packing ring, but without injury to the brass covering.

16. *Factors Influencing the Test Results.*—As previously stated, a number of uncertainties were inherent in the fluid pressure tests. Direct measurements were taken to determine the effect of certain factors, but for others the effect must be estimated by reference to known properties of similar materials. Four principal items will be considered: (a) the question as to when and why oil penetrated the brass cover, (b) the influence of the pressure and the friction at the packing rings and the effect of the brass cover on the external stresses applied to the specimen, (c) the character of the internal stress distribution in the specimens, and (d) the factors affecting the strength of the concrete in the two-dimensional tests as compared with the tests in simple compression.

(a) *Effect of the Penetration of Oil on Final Failure.*—In nearly all cases the specimens were broken into two parts at failure and the brass covering completely ruptured. It is of primary importance to know at what stage of the test the breaking of the covering took place, and what relation the penetration of oil into the specimen had to the failure. Obviously if oil penetrated the interior of the specimen, the large direct axial tension thus produced would be the decisive element in producing rupture. If such penetration of oil was a primary cause of rupture, the pressures that the specimens held before they broke would be less than the true strength of the specimens in two-dimensional compression.

The observations made during tests indicate that the penetration of oil into the specimens in practically all cases occurred only after the specimens had begun to fail. While the photographs of specimens after failure show oil at the edges of the break, it seems certain that the oil came in contact with these surfaces only after the specimens had failed. In several cases the specimens were broken in places where the brass covering did not break, and in these cases the broken surface was found to be free from oil when the covering was removed. It is clear that penetration of oil had no effect on the failure in these cases. It was also observed that the condition of the surface of the specimens adjacent to the break showed a local deformation widely different from that to be found at failure in axial tension. Particularly in the case of the leaner mixtures, the surface of the concrete was broken down around the fracture and the measurements of the diameter of specimens after the tests in all cases showed a reduction in diameter near the fractured surface. Other evidence is found in the load—deformation curves which show that the specimen generally deformed very rapidly before it finally broke. This is not characteristic of the stress-strain curve for tension with a brittle material such as concrete. Finally, the fact that the magnitude of the oil pressures at failure generally varied quite consistently with the strength of the concrete indicates that the penetration of oil could not regularly have been the cause of failure.

While it is thus quite certain that in the majority of cases the penetration of the oil was the result rather than the cause of failure, there were a few definite indications that penetration caused failure of certain specimens. Specimen A1 failed without any notable increase in the rate of deformation. An examination after failure showed a large cavity near the surface at the break where the brass had evidently been ruptured. Specimens A2, B1, B6, and D2, also showed some indication of irregularity which might be due to the penetration of oil.

(b) *Effect of Packing and Brass Covering.*—The intensity of the normal pressure between the packing and the specimen was probably always greater than the oil pressure on the specimen, at least at the inner edge of the packing. The tests show some effect of this excess pressure. On the 19 specimens without steel sleeves made from the 6 richer mixtures, 11 failed within the packing ring. All of the 6 specimens without sleeves made from the two leaner mixtures failed within the packing ring. It is seen from the results of the tests given in Table 4 that, with one exception, the average of the ultimate pres-

TABLE 4

PRINCIPAL RESULTS OF TESTS OF SERIES 2

Oil pressures not corrected for effect of end restraint of cylinder

Biaxial Compression 4 by 22-in. cyl. without sleeves					Simple Comp. 4 by 8-in. cyl.			Biaxial Compression 4 by 22-in. cyl. with sleeves					Simple Comp. 4 by 8-in. cyl.		
Spec. No.	Reduction in Diam. at Break in.	Oil Pressure lb. per sq. in.		Unit Load lb. per sq. in.		Spec. No.	Reduction in Diam. at Break in.	Oil Pressure lb. per sq. in.		Unit Load lb. per sq. in.					
		Critical Load*	Max. Load	Critical Load†	Max. Load			Critical Load*	Max. Load	Critical Load†	Max. Load				
Mixture A. 1:1 Mortar $w/c=0.56$															
A1		2890‡	3820	5250‡	A4	0.010	3600	3760			4030			
A2	0.003	4180	3580	3930	4900	A5	0.012	3620	3930			5250			
A3	0.011		4630	4560	A6	0.013	4150	4450			3400			
Av.			4100	4730	Av.		3790	4050			4230				
Mixture B. 1:2 Mortar $w/c=0.66$															
B1		2720	3300	3945	B4	0.015	3300	3730	2480		3110			
B2	0		2960	3000	3230	B5	0.008	3220	3410			3460			
B3	0		2960		3600	B6	0.004	4100	4100			4140			
Av.			2880		3590	Av.		3540	3750			3570			
Mixture C. 1:3 Mortar $w/c=0.94$															
C1	0.050		1690	1370	1670	C4	960	1300	1350		1620			
C2	0.007		1620	1670	1930	C5	0.023	1360	1720	1060		1250			
C3	0.008	1325	1695	1280	1700	C6	0.011	2140	2610	2160		2400			
Av.			1670	1440	1770	Av.		1490	1880	1520		1760			
Mixture D. 1:1:2 Concrete $w/c=0.64$															
D1		2890	3000	3480	D4	0.002	2890	2890			2860			
D2		2820	2330	2610	D5	0.006	2600	3230			2560			
D3	0.002	2600	2890		3060	D6	0.005	4120	4280	3320		3840			
Av.			2870		3050	Av.		3200	3470			3090			
Mixture E. 1:2.1:2.5 Concrete $w/c=0.88$															
E1	0.017		1970	1670	1975	E4	0.014	1310	1425	1250		1650			
E2	0.014		2030	1430	1680	E5	0.009	1710	1870	1720		2060			
E3	0.004	1520	1870	1600	1940	E6	0.004	1900	2610	1620		1990			
Av.			1960	1570	1870	Av.		1640	1970	1530		1900			
Mixture F. 1:2:4 Concrete $w/c=0.97$															
F3	0.017		1090	930	1230	F5	0.009	1430	2310	1180		1370			
F4	0.370		1350	930	1210	F6	0.019	1450	2210	930		1340			
F8	0.026	1320	2060	1700	1960	F7	0.019	1800	2110	1380		1595			
Av.			1380	1220	1470	Av.		1560	2210	1160		1400			
Mixture G. 1:3:5 Concrete $w/c=1.25$															
G1	0.096	¶	530	770	G4	0.038	570	1250	420		660			
G2	0.114		890¶	580	810	G5	0.010	900	1400§	610		840			
G3	0.132	700	1250¶	320	625	G6	0.031	1000	1770	720		840			
Av.				480	740	Av.		820	1510	580		780			
Mixture H. 1:4:7 Concrete $w/c=1.66$															
H3	0.214		360¶	500		H5		580§			345			
H4	0.076	¶	470		H7	0.047	750	1770			430			
H6	0.063	750	1200¶	250	380	H8	0.069	530	1770			390			
Av.				450		Av.		640	1770			390			

*Load at which slope of pressure-deformation curve was one-tenth the initial slope.

†Load at which slope of load-deformation curve was one-third the initial slope.

‡Omitted from average.

¶Pressure lost due to deformation and fracture within packing ring.

§Pressure lost; probable penetration of oil; omitted from average.

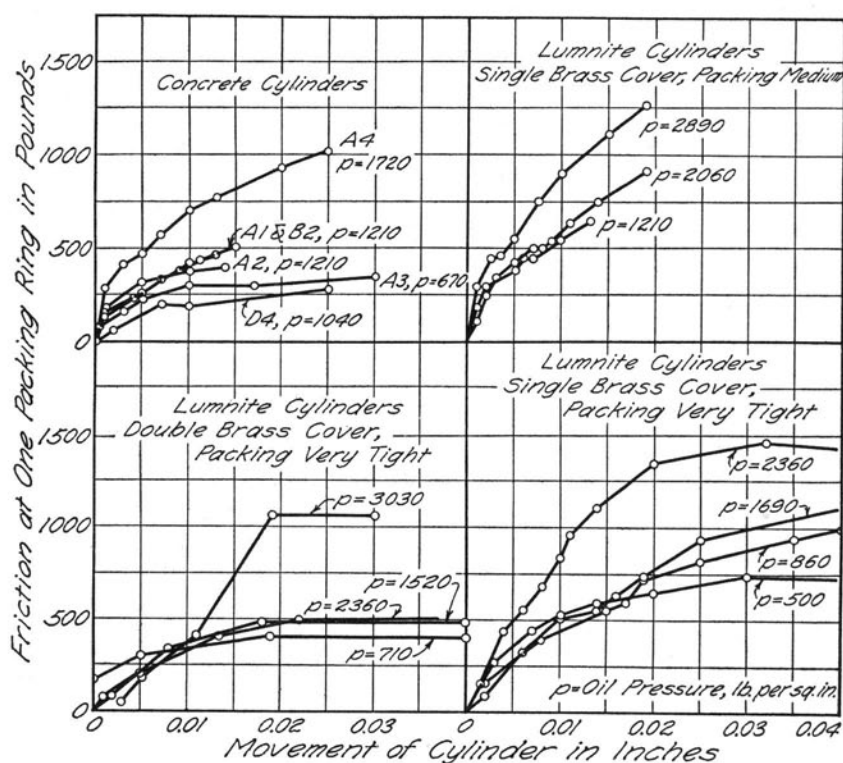


FIG. 7. RELATION BETWEEN FRICTION WITHIN PACKING RINGS AND MOVEMENT OF SPECIMEN

sures carried by the specimens without sleeves was lower than the average carried by the corresponding specimens with sleeves. The indications are that the effect of the packing was rather variable, and in later studies the specimens without sleeves are in many cases omitted from consideration.

Experiments were made to determine the friction exerted by the packing, as described in Section 14. The results of the friction tests on specimens with brass covering are given in Fig. 7. The friction found in individual cases generally varied somewhat with the amount of movement of the specimen, and reached a more or less constant value after a movement of 0.01 to 0.02 in. had taken place. Assuming that the friction at one packing ring was equal to one-half of the total load required to push the specimen through the chamber in the friction tests, and considering that the total elongation of the speci-

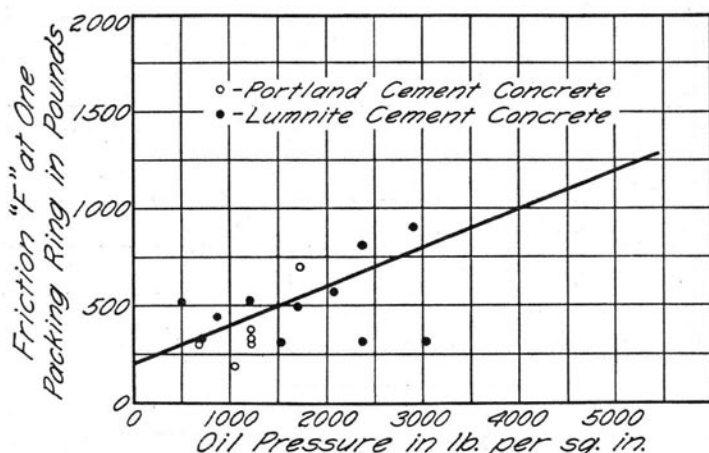


FIG. 8. FRICTION WITHIN PACKING RING AT MOVEMENT OF 0.01 IN.

men in the two-dimensional tests was 0.01 in. or less, Fig. 8 has been plotted showing the amount of friction in one ring at a movement of the specimens of 0.01 in. plotted as ordinates against various oil pressures as abscissas. While the points appear to be somewhat scattering, it seems reasonable to represent the average result by the straight line shown.

The unit elongation of the specimens at the critical pressure previously mentioned averaged about 0.001 in. per in. Considering the thickness and other properties of the brass, this would produce a total axial tension of about 350 pounds in the brass cover at the critical pressure. Tension tests of the brass showed the ultimate load on a strip one inch wide to be 42 pounds. When the specimens failed, the total tension in the brass may therefore have been as much as 500 pounds for specimens covered with one layer of brass, and twice as much for those having two layers.

The effect of the friction and of the tensile stress in the brass covering may be considered to be equivalent to that of an axial compression in the concrete. From the values found from the friction tests and the estimated value of the tension in the brass covering, the average total end pressures for various specimens at the critical pressure and at failure have been calculated and are listed in Table 5. It is seen that this end pressure or restraint amounted to from 80 to 165 lb. per sq. in. for the different specimens at the time of failure.

(c) *Distribution of Internal Stress in Specimens.*—The cylinders of Series 2 were broken by a uniform pressure applied to their sides

TABLE 5
ESTIMATED END RESTRAINT DUE TO FRICTION AND BRASS COVERING
OF SPECIMENS OF SERIES 2

Mixture		At Critical Oil Pressure				At Maximum Oil Pressure			
Ref. No.	Kind	Friction lb.	Re- straint of Cover- ing lb.	Combined Restraint		Friction lb.	Re- straint of Cover- ing lb.	Combined Restraint	
				lb.	lb. per sq. in.			lb.	lb. per sq. in.
A	1:1	950	700	1650	140	1000	1000	2000	165
B	1:2	870	580	1450	120	920	830	1750	145
C	1:3	500	350	850	70	550	500	1050	90
D	1:1:2	820	580	1400	115	870	830	1700	140
E	1:2. 1:2. 5	500	350	850	70	550	500	1050	90
F	1:2:4	500	350	850	70	550	500	1050	90
G	1:3:5	350	350	700	60	500	500	1000	85
H	1:4:7	300	350	650	55	450	500	950	85

along only a portion of their total length. The effect of such loading is to cause secondary axial stresses; an analysis of such stresses indicates that a secondary compressive stress is set up near the surface of the cylinder, while a corresponding tensile stress is set up in the interior of the cylinder. The compressive stress near the surface is probably larger than the tensile stress, and these stresses should be comparatively small over the middle half of the loaded portion of the specimen, and greatest near the ends of the loaded portion. However, at the higher loads these secondary axial stresses were probably relatively reduced by the inelastic deformation of the concrete. It may be concluded that at the oil pressures approaching those causing failure the stress over the larger part of the specimen was very close to that of pure two-dimensional compression, with the possibility that in regions near the end of the loaded portion the secondary stresses were of such magnitude as to have appreciable effect on the failure; an effect greater in the richer specimens, in which the concrete yielded less before failure, than in the leaner ones.

(d) *Factors Affecting the Strength of the Concrete in the Specimens for Simple and for Two-dimensional Compression Tests.*—While the concrete used in these tests had some variation in strength, as indicated by the test results of the 4 by 8-in. cylinders listed in Table 4, it may be assumed that the original quality of the concrete was the same for the simple as for the two-dimensional compression tests. Appreciable differences may have existed, however, in the drying for the two kinds of specimens. After removal from moist storage, the

4 by 8-in. cylinders were left in the air of the laboratory for 24 hours before testing, and may have dried out considerably. The 4 by 22-in. cylinders, on the other hand, were first given a surface treatment of Lumnite cement and plaster of Paris; then, about 6 to 8 hours after removal from the moist room, the brass covering was put on, thus effectually preventing further evaporation. It seems probable that the simple compression tests were made with a concrete somewhat stronger and stiffer than that used in the two-dimensional tests.

Another source of variation lies in the difference in the direction of the applied stresses relative to the direction of settlement when the concrete was poured. Tests have shown* that the strength of concrete when loaded in a direction perpendicular to the direction that was vertical during pouring, is considerably less than that when the concrete is loaded parallel to that direction. The adhesion between the mortar matrix and the coarse aggregate is weaker beneath the pebbles than elsewhere, and if the failure in compression is considered to be due to a splitting action, as will be discussed later, it seems logical that the strength should be less when these surfaces of weakness are parallel to the direction of loading than when they are normal to that direction.

17. *Discussion of Results.*—The principal results of the tests of Series 2 are given in Table 4, wherein the results of tests on specimens with and without sleeves are listed. The oil pressures given in the table are those actually observed during the tests, and have not been corrected to allow for the influence of end restraint of the test piece. Table 5 gives estimated values of the end restraint offered by friction at the ends of the specimen and by the strength of the brass covering. Table 6 gives a summary of the results of the tests of Series 2, in which the oil pressures have been corrected for the effect of the end restraint. This correction was made by the use of the results of the tests of Series 3B, in which the specimens received a lateral pressure greater than the end pressure. For specimens of Mixture E, the magnitude of the "critical" pressure, p_A , in Series 3B (determined as in Series 2) was found to be given approximately by the equation:

$$p_A = 1550 + 1.6 f_1$$

wherein 1550 represents the strength of the concrete in lb. per sq. in. in simple compression, and the last term represents the increase in the critical pressure produced by the end pressure or restraint, f_1 . For

*E. E. Butterfield, Discussion of Concrete Compression Test Specimens, Proc. A. S. T. M. Vol. 25, Part II, p. 251. 1925.

TABLE 6
SUMMARY OF RESULTS OF TESTS OF SERIES 2
Each value represents the average of three tests; only those specimens made
with sleeves have been included.

Mixture		Biaxial Compression Tests			Simple Compression Tests		Ratio $\frac{p'_A}{f_B}$
Ref. No.	Proportions	Oil Pressure at Critical Load		Initial Slope of Pressure Elongation Curve $\frac{E_2}{2\mu}$ lb. per sq. in.	Critical Unit Load f_B lb. per sq. in.	Initial Modulus of Elasticity E_1 lb. per sq. in.	
		Ob- served p_A lb. per sq. in.	Cor- rected p'_A lb. per sq. in.				
A	1:1	3970	3570	5 030 000	3590*	2 830 000	1.00
B	1:2	3540	3350	6 330 000	3060*	2 700 000	1.09
C	1:3	1490	1380	4 000 000	1520	1 880 000	0.91
D	1:1:2	3200	3020	6 300 000	2630*	3 070 000	1.15
E	1:2:1:2.5	1640	1530	4 070 000	1530	2 530 000	1.00
F	1:2:4	1560	1450	5 130 000	1160	1 970 000	1.25
G	1:3:5	820	720	3 070 000	580	1 480 000	1.24
H	1:4:7	640	550	1 770 000	330*	1.67

*Critical load estimated at 85 per cent of ultimate load.

the purpose of correcting the values of Table 6 it has been assumed that for all mixtures the critical pressure has been raised by an amount equal to 1.6 times the estimated value of the end restraint in pounds per square inch of the cross-section of the specimen.

The pressure—elongation curves for all specimens tested in biaxial compression are given in Fig. 9, and corresponding load—deformation curves for the companion specimens tested in simple compression are given in Fig. 10. The average values of the initial slopes of these curves are given in Table 6. A comparison of the slopes of the curves for the biaxial test specimens having sleeves with the corresponding values for the curves for the simple compression specimens shows that the initial slopes of the pressure—elongation curves are lower than would be expected. Considering the initial modulus of elasticity of the concrete to be the same in the two kinds of specimen, the slopes of the pressure—elongation curves correspond to values of Poisson's ratio of from 0.19 to 0.31, with no consistent variation in the values with richness of mixture. These values are so unusually high that they must be taken to indicate the effect of a difference in stiffness in the two kinds of specimen. As stated in Section 16, it appears that the specimens for the biaxial test had a greater moisture content than those tested in simple compression, with the result that both the stiffness and strength of the concrete in the 4 by 22-in. cylinders were considerably less than for the 4 by 8-in. cylin-

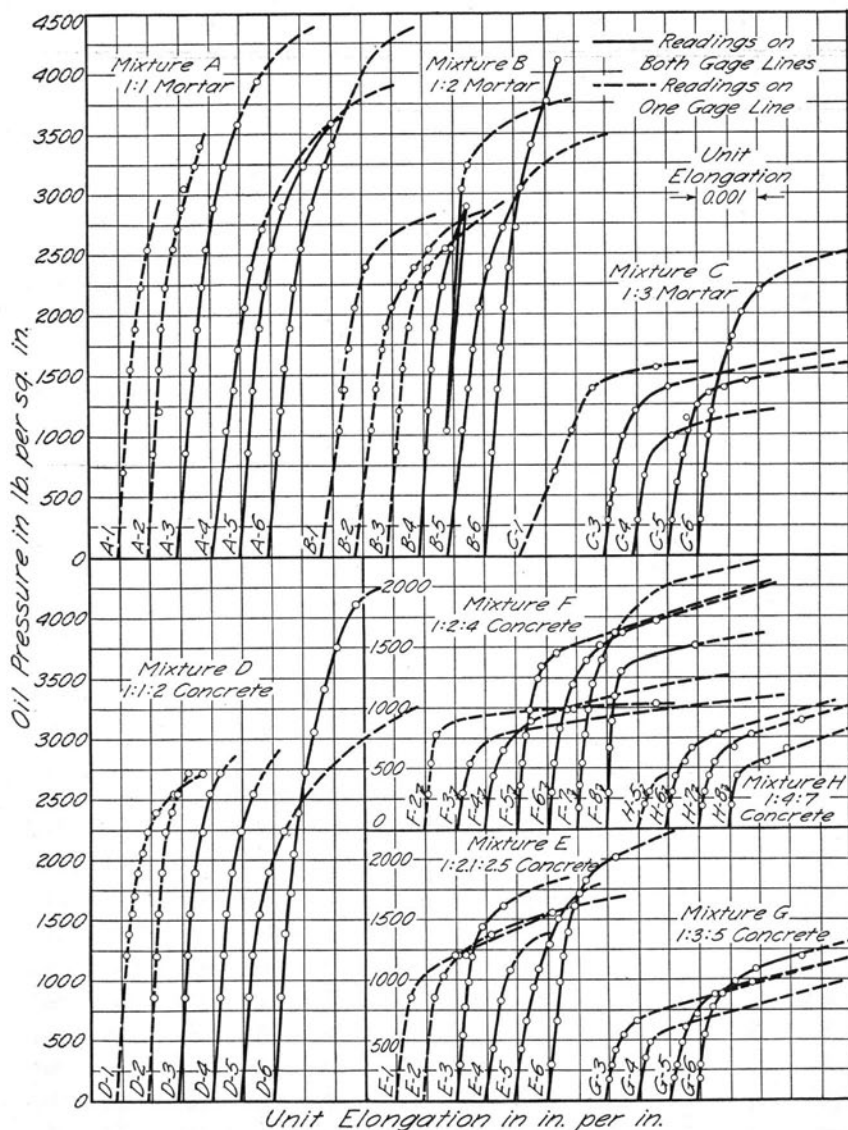


FIG. 9. PRESSURE-ELONGATION CURVES FOR SPECIMENS IN BIAxIAL COMPRESSION, SERIES 2

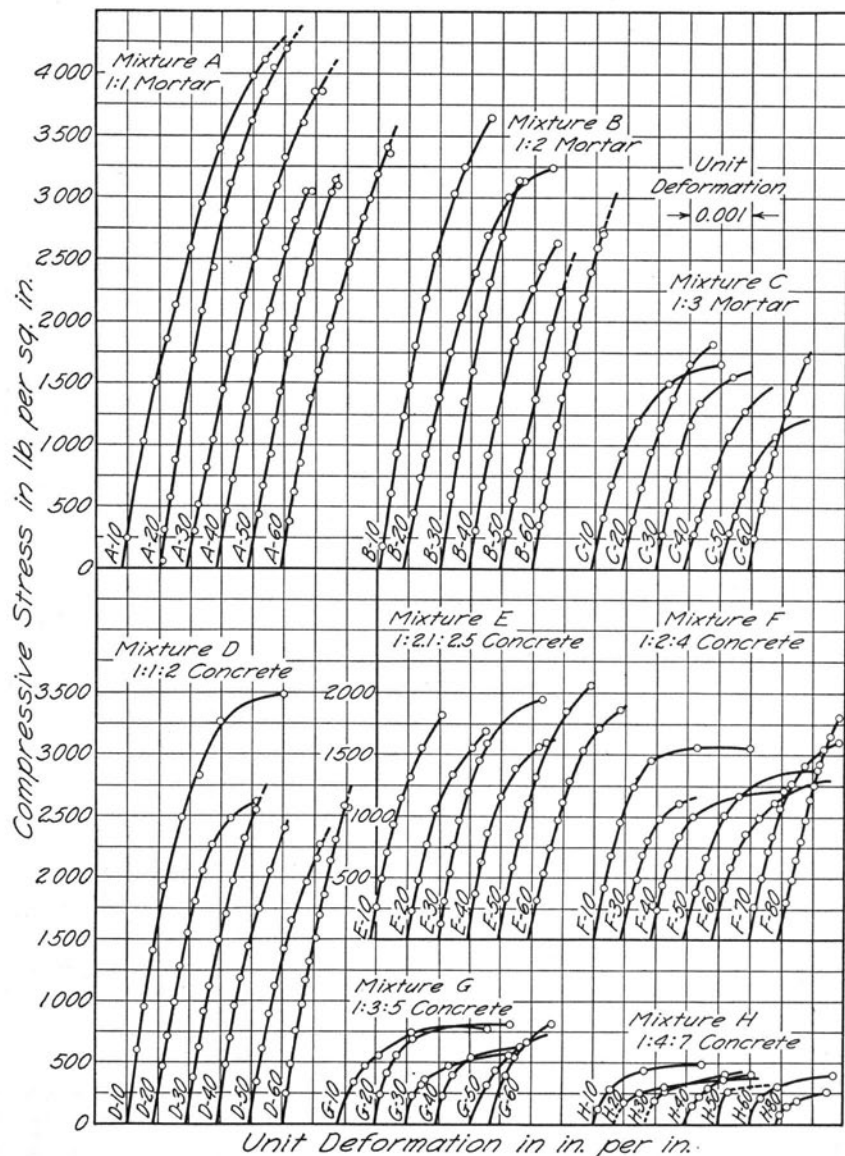


FIG. 10. LOAD-DEFORMATION CURVES FOR SPECIMENS IN SIMPLE COMPRESSION, SERIES 2

ders. It is possible also, considering the deformation of the concrete inside the steel sleeves, that the effective gage length of the 4 by 22-in. cylinders was somewhat greater than the ten inches used in computing unit deformations.

In comparing the two types of test, one having measurements of longitudinal deformations only and the other having measurements of lateral deformations only, it is convenient to refer to tests of plain concrete compression members in which both lateral and longitudinal deformations were measured. These tests were performed as a part of this investigation as Series 1 and will be described in a later bulletin. From measurements of longitudinal and lateral deformations that were made it was found that at loads of 75 to 85 per cent of the maximum the rate of lateral deformation suddenly increased, and finally caused the apparent bulk of the material to increase instead of to decrease with further application of load. In other words, the ratio of lateral to longitudinal deformation became greater than 0.5 at these high loads. This apparent dilatation indicates that splitting failure was beginning to develop at this point.

The pressure elongation curves of Fig. 9 are similar to the load—lateral-deformation curves for the specimens of Series 1, showing a decided increase in curvature at loads approaching the maximum. While in the case of the latter specimens, which were all of one mixture, the amount of load carried after the increase in volume had started was a fairly constant proportion of the maximum load, there were considerable differences in the added pressure that could be applied in the two-dimensional tests after the rate of axial elongation of the specimens had become large. For the rich mixtures this added pressure was quite small; for the lean ones, it was comparatively high. This may be explained by the fact that in the rich specimens visible cracks would be expected to develop which, with the high oil pressures used, would cause the failure of the brass covering earlier than in the case of the lean ones. Furthermore, in the lean specimens, it is quite possible that no definite cracking occurred, but that only a disorganization, or plastic flowing, of the material took place. The breakdown of the material thus might require considerable time for the material to flow in the direction of the length of the cylinder. It was apparent that time rather than load governed the deformation at failure of these lean specimens.

Table 4 gives information regarding the loads carried in the biaxial and simple compression tests. However, it is evident that the conditions of the tests do not warrant a direct comparison of ultimate

strengths. The presence in the two-dimensional tests of an oil pressure tending to rupture the brass envelope, and thus to precipitate failure in the stronger specimens, and the large difference in the deformations and in the ability of certain mixtures to undergo large plastic deformations, point to the need of some other criterion of strength than simply that of maximum load carried.

In selecting an intensity of pressure that may be used as a criterion of the resistance of the concrete, it seems natural to choose the pressure corresponding to the sharp bend in the pressure—elongation curves of Fig. 9 as a critical pressure, since, according to the conception of failure outlined in Section 6, the splitting and disorganization of the material starts at this point. Referring again to the tests of plain concrete compression members of Series 1, the critical load was reached when the average slope of the tangents to the stress—longitudinal-deformation curves was about one-third the average initial slope E , while the slope of similar curves for stress and lateral deformation was about one-tenth the initial slope, $\frac{E}{\mu}$. There is, of course, no certainty that the characteristics of the load—deformation curves at this critical point correspond exactly with similar ones for the tests of Series 2, in which several different mixtures were used, and in which the loads were applied in a different way. However, what experimental evidence is available indicates a similarity in the shape of the load—deformation curves from the two tests, and it seems proper to select as a “critical” point in the two-dimensional tests the pressure at which the slope of the pressure—elongation curves was one-tenth the initial slope, and as a “critical” point in the simple compression tests the stress at which the slope of the stress—longitudinal-deformation curves was one-third the initial value.

Points on the pressure—elongation curves of Fig. 9 have been located where the slopes of the tangents to the curves were one-tenth of the initial slope and the critical pressures at these points have been listed in Table 6, together with corresponding values from Fig. 10. It has been noted that the excess normal pressure existing within the packing ring caused the specimens without sleeves to fail at loads somewhat less than for those with sleeves. Accordingly, the following conclusions will be based entirely on the tests of the specimens which were provided with sleeves, generally three specimens of each mixture. In some cases, values of the critical stresses for specimens tested in simple compression were not determined, since the extensometer could not be left on the richer specimens till they failed, and for these

specimens an assumed stress equal to 85 per cent of the average ultimate strength for the mixture was taken for comparison with the critical pressure from the two-dimensional tests. It is felt that these assumed values are in fair agreement with the values determined experimentally. The ratio in Table 6 headed $\frac{p_A'}{f_B}$ gives a direct comparison

between the critical stresses for the different mixtures in two-dimensional compression and in simple compression. In all but one case, that of Mixture C, the critical stress in two-dimensional compression is as great as or greater than the critical stress in simple compression, although the difference is slight for all but the lean mixtures. In making this comparison it is to be remembered that the following factors mentioned in Section 16 have contributed to reduce the strength in the two-dimensional compression tests:

(1) Secondary stresses were induced in the 4 by 22-in. cylinder due to the fact that it was loaded over only a part of its length; these stresses probably were of greatest effect at the critical load since up to this point there had been little plastic flow, which would tend to relieve them at higher loads.

(2) A difference in drying out of the specimens probably caused the concrete in the two-dimensional tests to be weaker and less stiff than in the simple compression tests.

(3) The difference in direction of the applied stresses relative to the direction of settlement of the mixture during pouring probably gave the two-dimensional test specimens less resistance than those tested in simple compression.

It may be concluded that for the mixtures represented in these tests the strength of the concrete in two-dimensional compression was at least as great as the strength in simple compression. For the leanest mixtures, it was 25 to 50 per cent higher, and the circumstances of the tests indicate that for the richer mixtures it would also have been found to be higher had the conditions of the tests been perfectly alike in the two cases.

IV. CONCRETE IN TRIAXIAL COMPRESSION, SERIES 3A

18. *Purpose of Three-dimensional Tests.*—The direct object of these tests was to study the action of concrete specimens under compressive stresses in three directions, so applied that the magnitude of the stresses could be definitely controlled and varied, and some infor-

mation could be obtained regarding the deformation that occurred. As in Series 2, the tests of the following series had a dual purpose; they were intended to give information regarding the way in which the material breaks down under compression, and they were also planned to furnish information on the strength of concrete under compression when simultaneously subjected to compressive stresses in the other two principal directions.

The method used was that of applying a liquid pressure to the sides of a cylinder which was loaded in the axial direction in a testing machine. This method of testing limited the combinations of stress to the case in which two of the principal stresses were always equal. In order to secure a wide range in stress combinations, two groups of tests were devised. In one group, Series 3A, the two principal stresses produced by liquid pressure were smaller than the third principal stress produced by the end load. In the other group, Series 3B, the two equal principal stresses were greater than the third principal stress. Within each of these groups, a larger number of stress combinations could be produced in any specimen, since both the liquid pressures, and the end load could be varied within the limits of the strength of the specimens, and the capacity of the apparatus. The system was adopted of keeping the smaller principal stress or stresses constant throughout any individual test, and increasing the larger principal stress or stresses progressively until failure occurred. The magnitude of the smaller principal stress was varied systematically from one set of specimens to another.

19. *Outline of Series 3A.*—Series 3A included tests of sixty-four 4 by 8-in. concrete cylinders in three-dimensional compression, the two principal stresses normal to the axis of the cylinder being smaller than the axial stress. For comparison, sixteen 4 by 8-in. cylinders were tested in simple compression. Eight nominal intensities of oil pressure were used: 250, 500, 750, 1000, 1500, 2000, 3000, and 4000 pounds per square inch. Three different mixtures numbered 1, 2, and 3 were used; these mixtures were identical with Mixtures E (1:2.1:2.5), D (1:1:2), and G (1:3:5) used in Series 2. The 80 test cylinders were made in eight days, ten specimens per day, consisting of eight to be tested in three-dimensional compression, and two in simple compression. The arrangement of specimens was such that the tests made at any one value of the oil pressure included specimens from 2 or 4 different batches made on different days. The method of making the specimens was similar to that for Series 2 except that the ends of all specimens were capped with neat cement. The specimens were stored

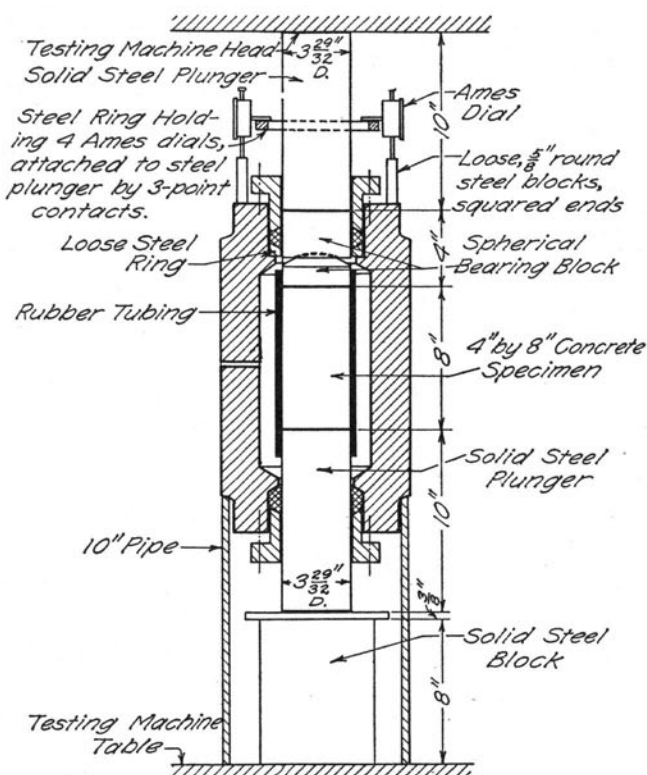


FIG. 11. ARRANGEMENT OF APPARATUS FOR TESTS OF SERIES 3A

in the moist room until the day before the test, when they were removed for surface treatment.

20. *Preparation of Test Specimens.*—The method used to protect the specimens against penetration of oil during the test was to enclose them inside a rubber tube. The success of the method depended upon the smoothness of the specimen, since the edges of small holes or cavities in the surface quickly punctured the tubing. To prevent this, a systematic treatment was given each specimen by rubbing the surface thoroughly with a paste of plaster of Paris until all surface cavities were filled.

After the specimen had been surface treated, gage lines for measurements of permanent deformation were laid out. Four longitudinal lines, 5 in. long, and 6 diametrical gage lines were used. Measurements were made before and after tests, those on longitudinal

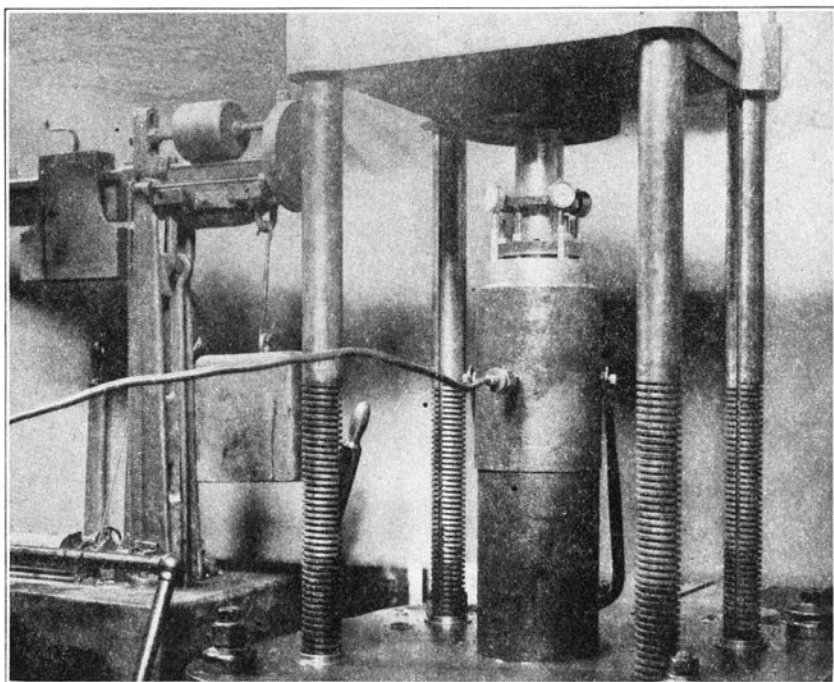


FIG. 12. TESTING APPARATUS AS USED IN SERIES 3A

lines with a steel scale, and those on diameters with a screw micrometer.

21. *Test Apparatus and Procedure.*—The arrangements of apparatus for the tests of Series 3A are seen from Figs. 11 and 12. The pressure chamber was the one used in Series 2. The 4 by 8-in. concrete specimen was placed in the pressure chamber between two solid steel plungers $3\frac{2}{3}\frac{1}{2}$ in. in diameter and 10 in. long, which extended out through the end openings of the chamber. Between one of the plungers and the specimen a spherical bearing block was used. The pressure chamber with the specimen in place was supported vertically on a piece of 10-inch pipe, and placed in the 300 000-lb. Olsen testing machine. The head of the testing machine was brought down on to the upper steel plunger, and the specimen with the two plungers was pushed downward through the pressure chamber until the bottom plunger came to bearing upon a solid steel block placed on the weighing table of the machine. With further movement of the head of the machine, applied at the rate of 0.05 in. per min., vertical load was ap-

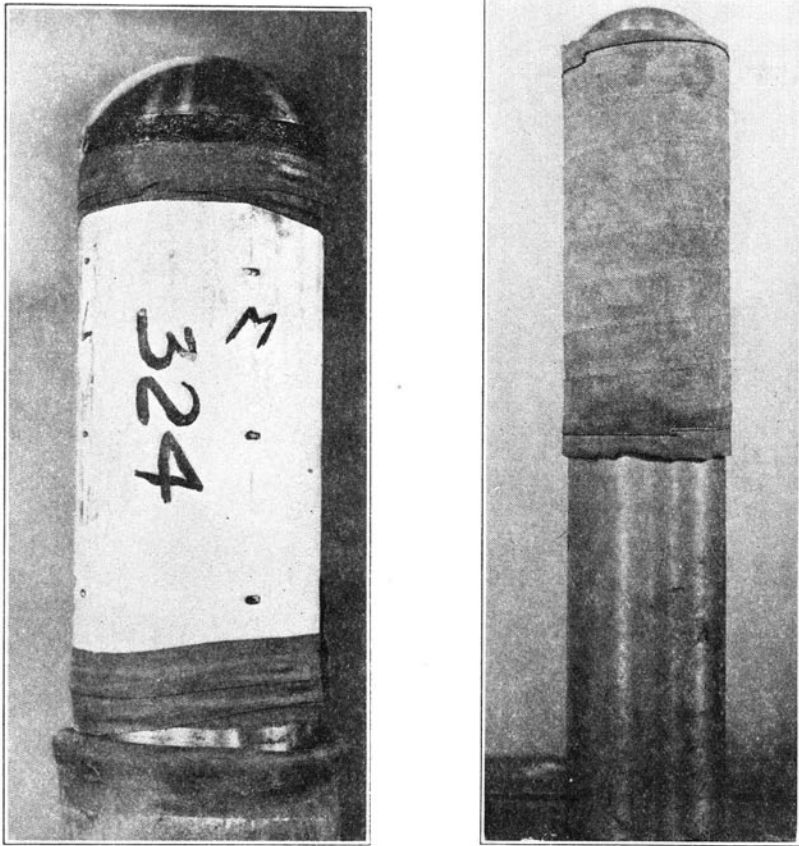


FIG. 13. VIEWS SHOWING METHOD OF ALIGNING AND PROTECTING SPECIMENS, SERIES 3A

plied to the specimen. Lateral oil pressure was applied by a hand pump as in Series 2.

The vertical movement of the upper plunger relative to the pressure chamber was measured by a set of Ames dials, as shown in Figs. 11 and 12. The dials were mounted on a steel ring attached to the upper plunger, and acted against four small steel blocks on the top surface of the pressure chamber. As previously noted, the specimen was protected against the penetration of oil by a piece of 6-in. rubber tubing. Care was taken in aligning the specimen, attaching it to the bottom steel plunger and to the spherical bearing block and enclosing it in the rubber tubing. The method used is illustrated in Fig. 13. The junction of the specimen and steel plunger was held in alignment

by several layers of friction tape. A similar joint was made between the specimen and the bearing block, the rubber tubing was then pulled up over the specimen, and the edges of the tubing were bound tightly against the steel surfaces by a wrapping of fine wire. A rubber cement was also applied to the steel surfaces under the edges of the tubing to insure a tight joint. With the pressure chamber lying horizontal, the specimen was slid into place on a wooden track, the top plunger was inserted, and the packing glands were brought into position. With this technique of protecting the specimen and inserting it in the chamber, definite assurance of effective oil proofing of the specimen was secured.

In starting a test the specimen was subjected to a small initial end load and the oil pressure was then applied. Since the oil had access to the spherical bearing block it produced a condition of equal pressure on all sides of the specimen. In addition, the head of the testing machine was lowered sufficiently to apply a small excess end pressure at all times. When the oil pressure had been raised to its predetermined value, the Ames dial apparatus was adjusted and initial deformation readings were taken. The axial unit load at these "zero" readings was slightly higher than the oil pressure. The testing was continued by applying the end load in increments (varying with the oil pressure used) of 5000 to 20 000 lb. The testing machine was stopped between increments for three or four of the lower loads; thereafter simultaneous readings of load and deformation were taken with the machine running continuously, in order to minimize the effect of time upon the deformation. The test was usually continued until the load on the specimen had dropped off appreciably from the maximum value. After the specimen had been removed from the pressure chamber it was again measured over the gage lines previously described.

The simple compression tests were made as in the corresponding tests of Series 2.

Two series of friction tests were made, similar to those of Section 14. In the first test, the pressure chamber was set up as in the three-dimensional tests of Series 3A, but with no concrete specimen inside the chamber. With a constant oil pressure maintained, the top plunger was pushed down into the chamber against the oil pressure and the usual readings of movement of the plunger were taken. Under these conditions, the difference between the loads read on the beam of the testing machine and the total oil pressure over the cross-section of the plunger was necessarily equal to the friction between the pack-

ing and the steel plunger. The test was repeated for a number of oil pressures. In the second test an arrangement similar to the friction test of Series 2 was used. Instead of the brass-covered concrete specimen, however, a solid steel plunger $3\frac{29}{32}$ in. in diameter and 28 in. long was used. The surface of this plunger was machined to the same finish as that of the shorter ones previously used. Movements of the plunger were read by means of Ames dials as in Series 3A. The friction tests were made with different sets of packing which should represent fairly well the packing used in Series 3A.

The method of measuring deformations in Series 3A (see Fig. 12) was to measure the relative motion between the steel plunger and the pressure chamber. Assuming that there was no motion of the pressure chamber itself, the motion measured would represent the total deformation of all the parts from the point where the dials were attached to the upper plunger down to the table of the testing machine, including the deformation in the steel plungers, the spherical block, the 8-in. steel block at the bottom, and the specimen. A few tests were made in which all deformations except that of the specimen could be measured. The arrangement used for these tests was the same as that used in the regular tests of Series 3A, except that the concrete specimen was omitted and the spherical block was fastened by tape directly to the lower plunger. The 28-in. steel cylinder was used above the upper plunger and the Ames dial attachment was fastened to this cylinder, so that eight additional inches of steel cylinder were included in the gage length. Several tests were made with different oil pressures and the test was repeated at different times.

22. Precision of Measurements.—In Section 11 an estimate was given of the accuracy with which the oil pressure was measured. This estimate is believed to apply to the tests of Series 3A even though it was necessary to maintain pressures as long as 30 minutes. The pressure fluctuated during such tests, due to the lateral bulging of the specimens under load; the increase so produced was compensated for by releasing the pressure slightly through a valve in the pump. The pressure probably did not vary more than 50 to 100 pounds per square inch from the designed pressure in any test, so that the general course of the pressure—deformation curve was apparently not greatly influenced by these fluctuations.

The axial loads were applied to the specimens by the 300 000-lb. testing machine which, when calibrated against an Amsler standardizing box, was found to have a mean error of 0.6 per cent. The end

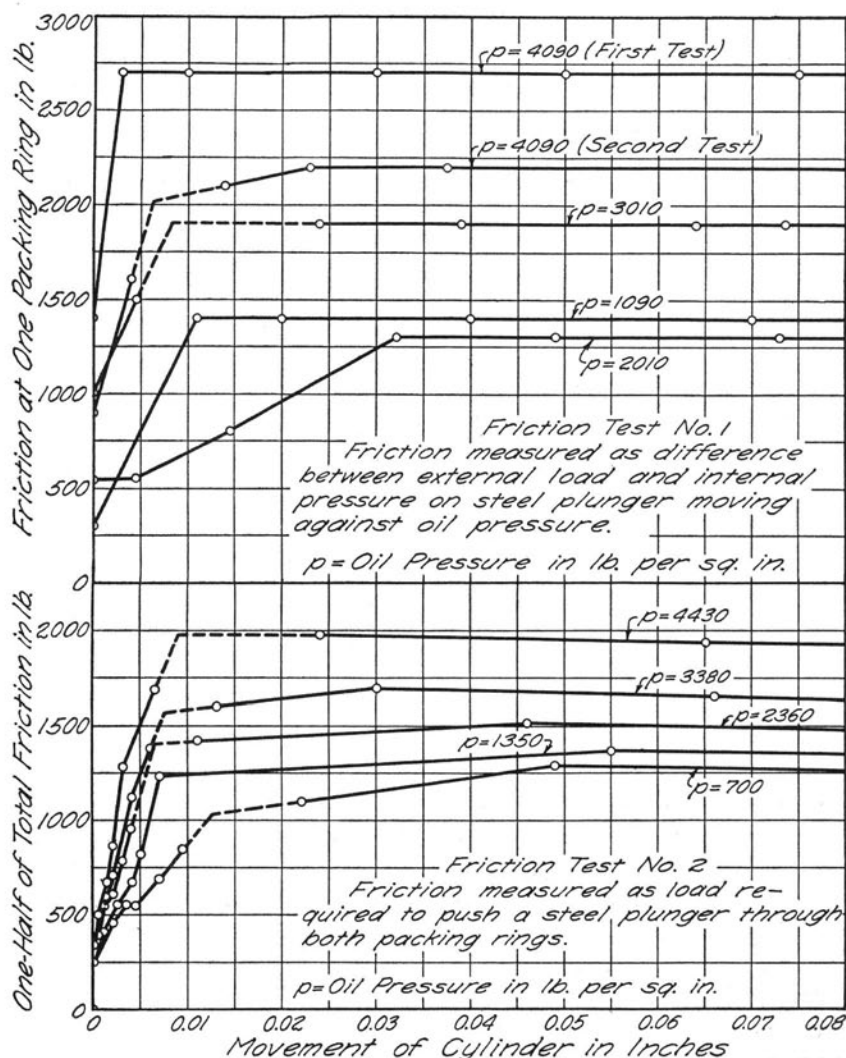


FIG. 14. RELATION BETWEEN FRICTION WITHIN PACKING RING AND MOVEMENT OF STEEL PLUNGER

load applied by the machine less the frictional resistance at one packing ring gives the net axial load on the specimen. The amount of the friction was determined by two methods as described in Section 21, and the results are given in Fig. 14, which shows that the friction was relatively small, and that it varied with the oil pressure and with the amount of movement of the plungers. The first method gave the fric-

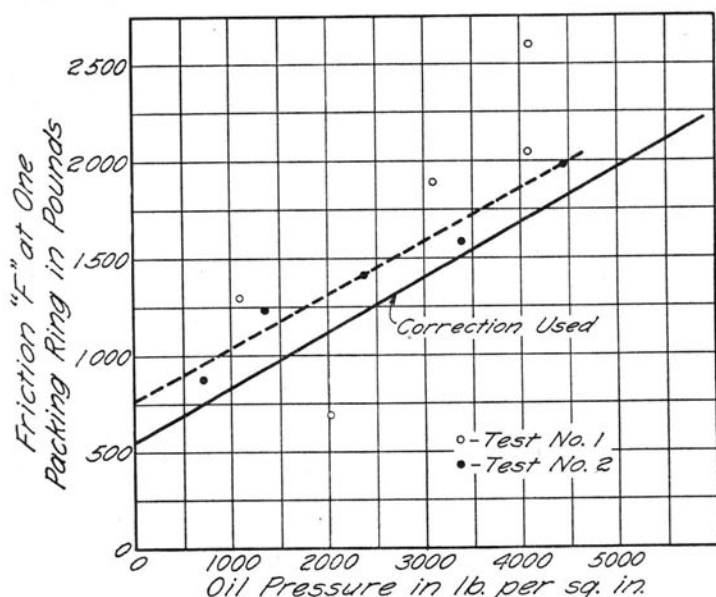


FIG. 15. FRICTION WITHIN PACKING RING AT MOVEMENT OF 0.01 IN.

tion as a difference between the load on the testing machine and the total oil pressure acting on the steel plunger, hence it does not seem as accurate as the second method in which the total frictional resistance of two joints was weighed on 10 000-lb. scales. Hence the main reliance was placed on the results of the second test in plotting Fig. 15, which shows the amount of friction observed at a movement of 0.01 in. at various oil pressures. This amount of movement is comparable to that produced in the three-dimensional tests. The heavy curve of Fig. 15, which gives values slightly less than the average of the observed values, was used in correcting the end loads in computing the maximum principal stresses in the test specimens.

The method of measuring deformations of the specimens in Series 3A was not as accurate as could be desired. It was hoped that at least the deformations taking place within the pressure chamber could be determined with sufficient accuracy (1) to indicate any important changes in the magnitude and in the rate of deformation throughout a test, and (2) to furnish a comparison of the rate of deformation of specimens subjected to different oil pressures. All deformation measurements were somewhat obscured by the defor-

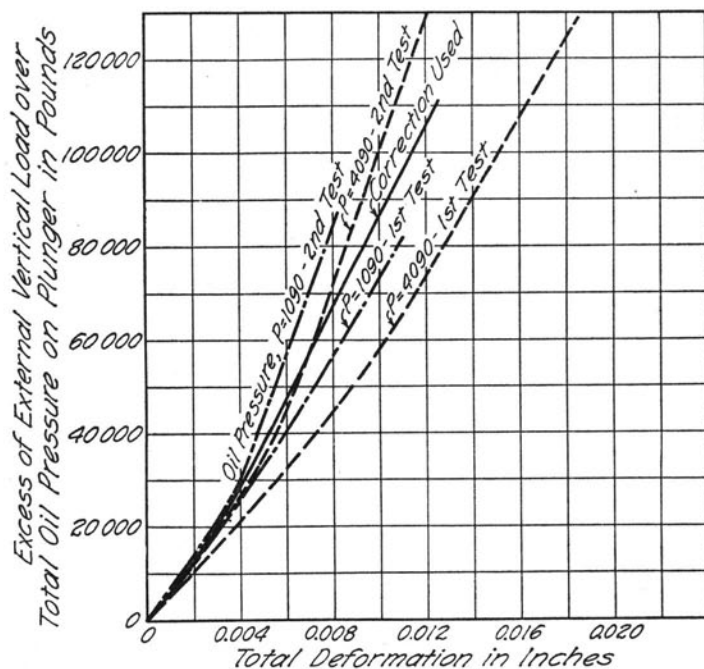


FIG. 16. LOAD-DEFORMATION CURVES FOR TESTING RIG ALONE

mations of the testing apparatus outside of the specimen; as noted in Section 21 an effort was made experimentally to evaluate this particular quantity. Typical load—deformation curves from the tests in which the deformation of the concrete specimen was eliminated are shown in Fig. 16. The curves shown represent the range of pressures used as well as the effect of differences in the assembly of the testing apparatus. Dial movements (see Fig. 12) are plotted against the excess of the vertical load over the total oil pressure acting on the end projection of the spherical block, corresponding to the portion of load for which deformations were measured on the concrete specimens. Evidently the oil pressure had little effect on the deformation; except for some rather large values at the lower loads, the deformations agreed fairly well with elastic deformations calculated from the known loads and the modulus of elasticity of the steel parts. The heavy solid curve of Fig. 16 represents a fair average of all of the tests made, and gives the correction that was subtracted from the deformations measured in this series in arriving at the deformations of the concrete speci-

mens. It may be concluded that the accuracy of deformation measurements is dependent upon the accuracy of the corrections applied to compensate for the "apparatus-deformations," since the possible error due to this cause was considerably more than the error of observation in the average of four Ames dial readings. It appears that the error produced in applying the correction may have amounted to as much as from 8 to 10 per cent of the total deformation in extreme cases.

23. *Phenomena of Tests.*—The only phenomena to be observed during the tests were the variations in load and the deformations. Some general characteristics of the tests may be mentioned. The dial readings at the lower loads showed a relatively high rate of movement of the top plunger with increase in load. This large deformation probably was related to a surface depression or a horizontal displacement within the spherical block. As might be expected, the deformations at early loads were quite irregular; hence the zero point on the load—deformation curves cannot be taken directly from the observed data at the lowest loads. As the load increased, the rate of deformation lessened, and in most tests became nearly constant at an excess load of from 10 000 to 20 000 lb., increasing again at higher loads until the maximum was reached. Specimens of different mixtures behaved somewhat differently in this respect; the gradual increase in the rate of deformation at the higher loads took place more slowly in the leaner mixtures than in the rich ones.

In the tests in which the machine was stopped for each set of readings the load fell off or released itself somewhat while readings were being taken. This release began at very low loads. The amount of the release of load increased as the load increased, as will be discussed later. At a load somewhat higher than that at which the releasing was first observed, an appreciable yielding or increase in deformation was noted during the time in which the machine was stopped. This yielding also increased as the maximum load was approached. For most of the tests the machine was run almost continually during the last part of the loading, and in practically all cases a point was finally reached at which the load remained nearly constant for a considerable time with the machine running. This was continued until the range of the dials was exceeded, and the test was discontinued, or until a definite decrease in load was seen. The specimens of the richer mixtures and those tested at the low oil pressures showed such a decrease in load most distinctly. In a few specimens, especially those from the leanest mixture tested at the highest oil pressures, the load kept increasing slowly until the range of the dials

was exceeded. Thus none of the four specimens of Mixture No. 3 tested at an oil pressure of 4090 lb. per sq. in. reached a maximum load within the range of movement of the dials, which was about one-half inch. In the test of one of these specimens, the machine was kept running for $1\frac{3}{4}$ minutes after the range of the dials had been exceeded; finally the maximum load was reached with a total shortening of the 8-in. specimen of about 0.65 in.

24. *Results of Tests.*—Load—deformation curves from the tests of Series 3A are given in Figs. 17 and 18, in which axial unit stresses are plotted as ordinates and axial unit deformations as abscissas, both stresses and deformations having been corrected as noted in Section 22. The curves are grouped according to the concrete mixture and the lateral oil pressure used. In each curve, a straight line is shown representing the tangent to the curve at low loads. The location of this tangent was made rather difficult by the irregularity of the plotted points in this region. In drawing the tangent a uniform method was used whereby the upper part of the curve where the points were quite regular was first drawn; the curve was then continued downward following the general trend of the lower plotted points, and the tangent was drawn to fit the curve thus determined. Figure 19 shows load—deformation curves for the simple compression tests of cylinders of the three mixtures. Figure 20 shows the recorded decrease in the load applied by the testing machine during the time of taking readings on a number of typical specimens. This decrease was evidently due to the deformation of the specimen, which continued after the testing machine had stopped. Table 7 contains the principal results of the tests of Series 3A. Regarding the items listed in the table it may be noted that the “Initial Slope of Load—Deformation Curves” is the slope at a load slightly greater than the oil pressure at which the first measurements of deformation were taken. The “Average Permanent Deformations” are the average of the deformations measured on four longitudinal and six lateral gage lines marked on the surface of each specimen.

Views of specimens after testing are given in Fig. 21. In each group of six cylinders, two specimens that had not been tested are included to indicate the changes in size and shape of the specimens produced by the tests.

25. *Effect of Lateral Pressures on Maximum Load.*—The maximum load reached with each specimen is listed in Table 7. It should be noted that in a few tests the load had not shown definite signs of

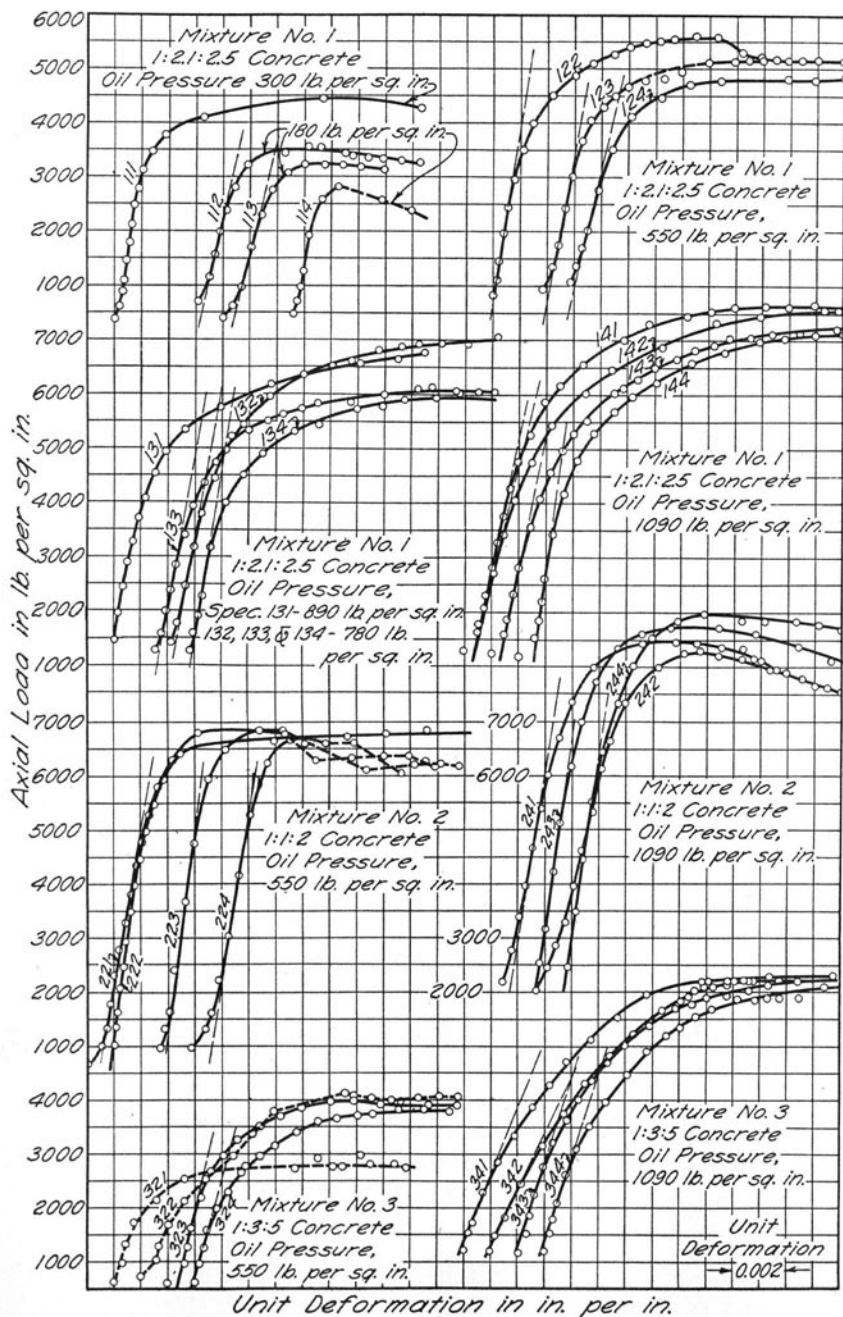


FIG. 17. LOAD-DEFORMATION CURVES FOR SPECIMENS IN TRIAXIAL COMPRESSION, SERIES 3A

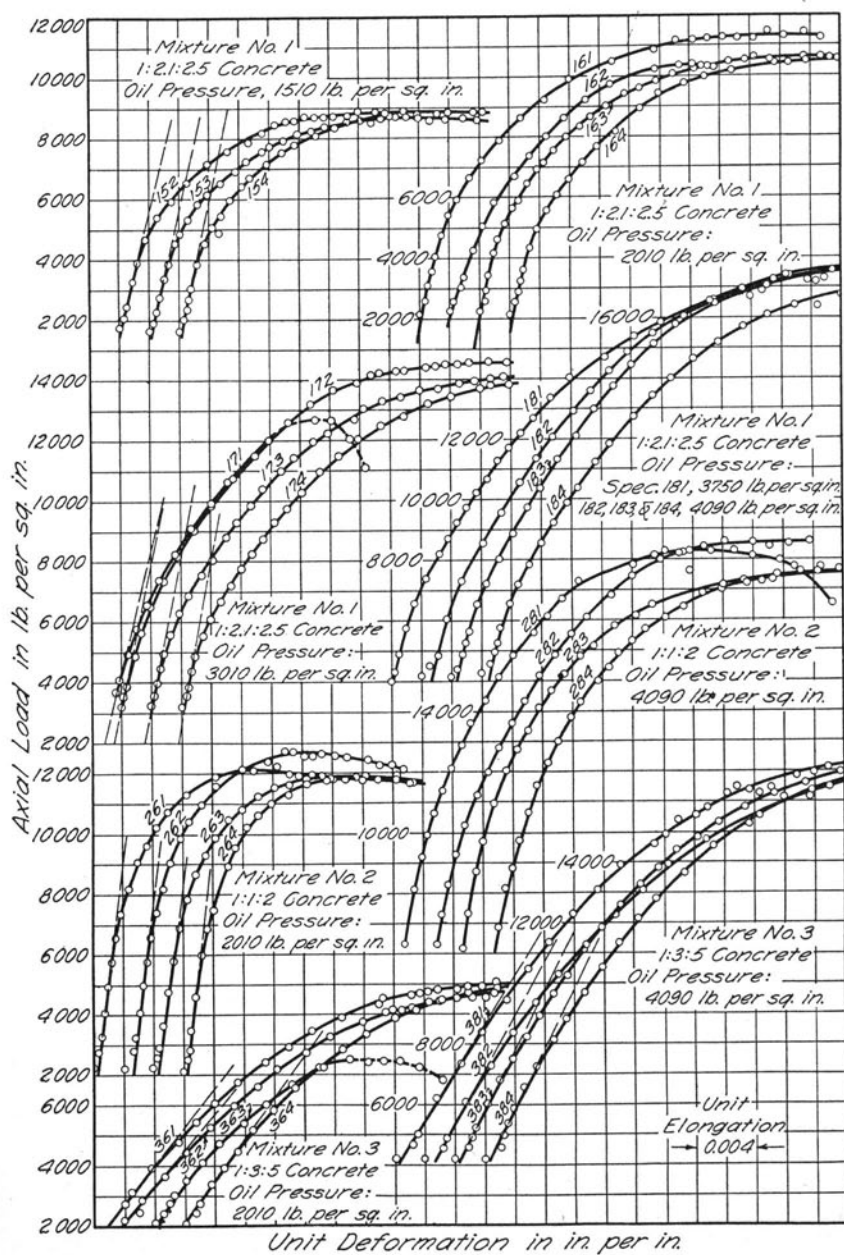


FIG. 18. LOAD-DEFORMATION CURVES FOR SPECIMENS IN TRIAXIAL COMPRESSION, SERIES 3A

TABLE 7
PRINCIPAL RESULTS OF TESTS OF SERIES 3A

Spec. No.	Oil Pressure lb. per sq. in.	Maximum Axial Unit Load lb. per sq. in.	Permanent Unit Deformation		Initial Slope of Load-Deformation Curve 1000 lb. per sq. in.	Spec. No.	Oil Pressure lb. per sq. in.	Maximum Axial Unit Load lb. per sq. in.	Initial Slope of Load-Deformation Curve 1000 lb. per sq. in.	Permanent Unit Deformation	
			Axial	Lateral						Axial	Lateral
Mixture No. 1. 1:2. 1:2. 5 Concrete $f'_c = 2575$ lb. per sq. in.											
111	300†	4450†	1800†	0.0171	0.0102	151	1510	9210	1200	0.0437	0.0125
112	180	3590	1100	0.0196	0.0107	152	1510	8900*	1200	0.0512	0.0182
113	180	3240*	1050	0.0098	0.0079	153	1510	8710	1100††
114	180	2840*	1500§§	154	1510	8880	1450	0.0650	0.0205
Av.		3420	1220§		Av.		8930	1230		
121	550	4980†	1600	0.0261	0.0080	161	2010	11 500	1300	0.0497	0.0200
122	550	5620	1300	0.0194	0.0208	162	2010	10 390*	750†	0.0306	0.0110
123	550	5190	1300	0.0235	0.0156	163	2010	10 650	1300	0.0532	0.0190
124	550	4910	1100	0.0440	0.0205	164	2010	10 550	1150	0.0611	0.0185
Av.		5240	1330			Av.		10 800	1250		
131	890†	6785†	1400†	0.0179	0.0082	171	3010	12 700†	1000§§
132	780	7090	1450	0.0364	0.0164	172	3010	14 600*	1150	0.0522	0.0169
133	780	6150	1450	0.0317	0.0114	173	3010	14 100	1250	0.0526	0.0202
134	780	6080	1600	0.0438	0.0161	174	3010	14 080	1350	0.0592	0.0190
Av.		6440	1500			Av.		14 260	1190		
141	1090	7665*	1200	0.0220	0.0059	181	3750†	17 300†	1050	0.0446	0.0154
142	1090	7650	1150	0.0348	0.0162	182	4090	17 750	1050	0.0528	0.0169
143	1090	7360	1250	0.0610	0.0246	183	4090	17 650†	1150††
144	1090	7170†	1850	0.0538	0.0233	184	4090	17 600	1250	0.0712	0.0202
Av.		7560	1360			Av.		17 670	1150		

*Oil penetrated rubber tubing during test.

†Not recorded.

‡Omitted from average.

§Maximum load not reached.

§Cylinder broken.

TABLE 7 (Continued)

Spec. No.	Oil Pressure lb. per sq. in.	Maximum Axial Unit Load lb. per sq. in.	Permanent Unit Deformation		Initial Slope of Load-Deformation Curve 1000 lb. per sq. in.	Spec. No.	Oil Pressure lb. per sq. in.	Maximum Axial Unit Load lb. per sq. in.	Initial Slope of Load-Deformation Curve 1000 lb. per sq. in.	Permanent Unit Deformation	
			Axial	Lateral						Axial	Lateral
Mixture No. 2. 1:1:2 Concrete $f'_c = 3660$ lb. per sq. in.											
221	550	6800	0.0315	0.0192	1450	321	550	2900†	...	0.0556§	0.0230§
222	550	6820	0.0316	0.0169	1900	322	550	4240	...	0.0556§	0.0230§
223	550	6830	0.0278	0.0159	1900	323	550	3980	1000	0.0620§	0.0225§
224	550	6630	0.0316	0.0187	1550	324	550	3840	900		
Av.		6770			1700	Av.		4020	950		
241	1090	8470	0.0198	0.0105	2100	341	1090	6290	600	0.0652	0.0141
242	1090	8300	0.0277	0.0068	...	342	1090	6320	500	0.0356	0.0164
243	1090	8730	0.0297	0.0123	2300	343	1090	6300	800	0.0658	0.0172
244	1090	8990	0.0397	0.0195	2100	344	1090	6120	750	0.0558	0.0123
Av.		8620			2170	Av.		6260	660		
261	2010	12 140	0.0296	0.0102	2000	361	2010	10 180	325	0.0534	0.0198
262	2010	12 750*	0.0318	0.0146	2000	362	2010	10 050	300	0.0594	0.0195
263	2010	11 820	0.0568	0.0185	1800	363	2010	7 500†	...	0.0594	0.0195
264	2010	11 900	0.0475	0.0180	2200	364	2010	10 300	350	0.0594	0.0195
Av.		12 150			2000	Av.		10 175	325		
281	4090	19 300*	0.0439	0.0159	1900	381	4090	17 250†	400	0.0536	0.0202
282	4090	19 600*	0.0668	0.0195	1500	382	4090	17 150†	450	0.0696	0.0195
283	4090	19 000	0.0614	0.0172	2100	383	4090	17 850	475	0.0696	0.0195
284	4090	18 750			2100	384	4090	17 500†	450	0.0696	0.0195
Av.		19 160			1900	Av.		17 440	445		
						384	6560	24 600†		0.0780	0.0200

*Oil penetrated rubber tubing during test.

†Omitted from average.

*Maximum load not reached.

§Cylinder broken.

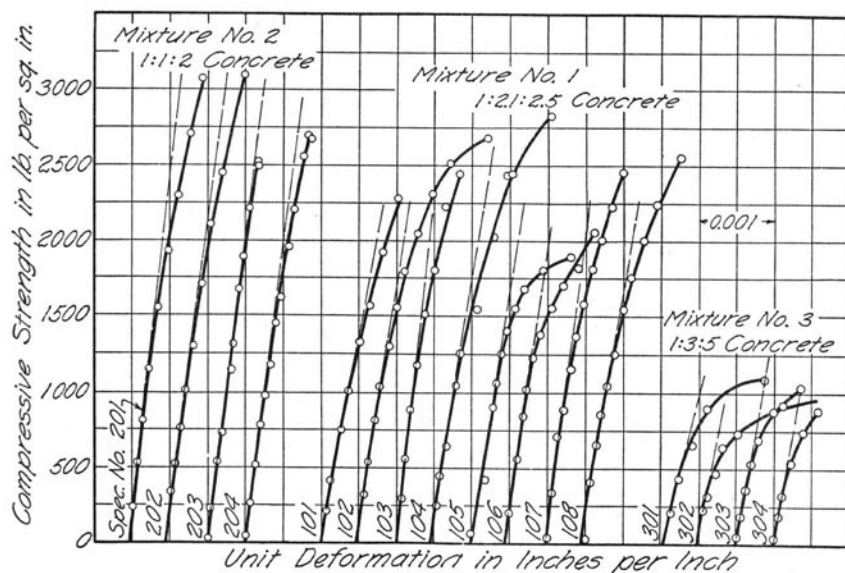


FIG. 19. LOAD-DEFORMATION CURVES FOR SPECIMENS IN SIMPLE COMPRESSION, SERIES 3A

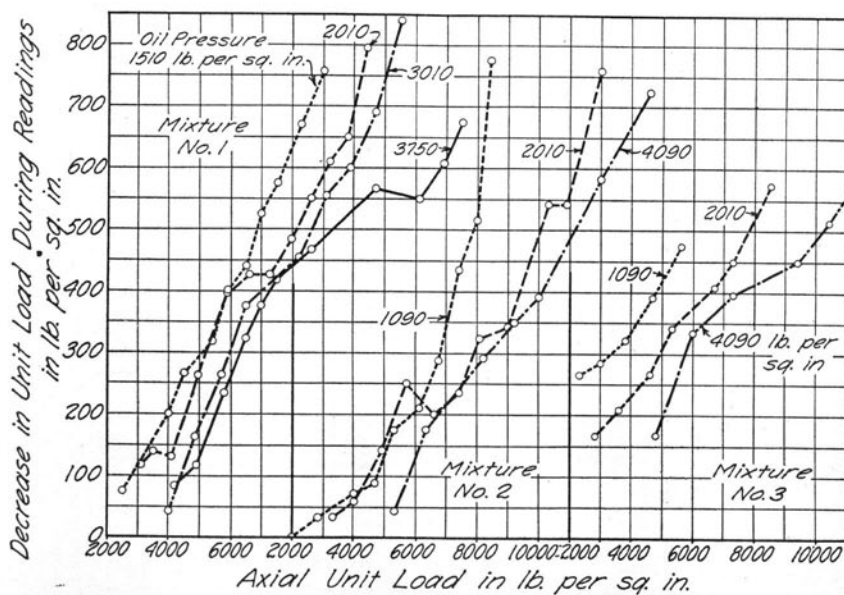


FIG. 20. OBSERVED DECREASE IN LOAD DURING OBSERVATIONS



FIG. 21. SPECIMENS OF SERIES 3A, AFTER TEST

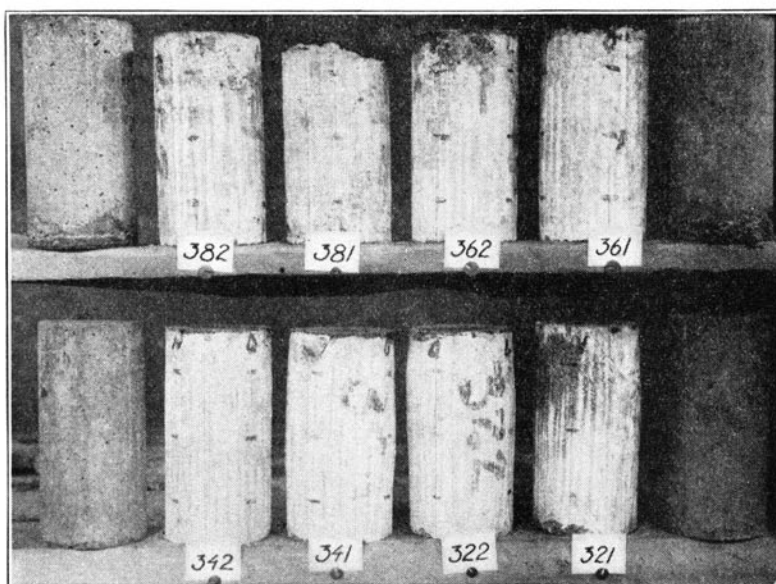


FIG. 21 (CONTINUED). SPECIMENS OF SERIES 3A, AFTER TEST

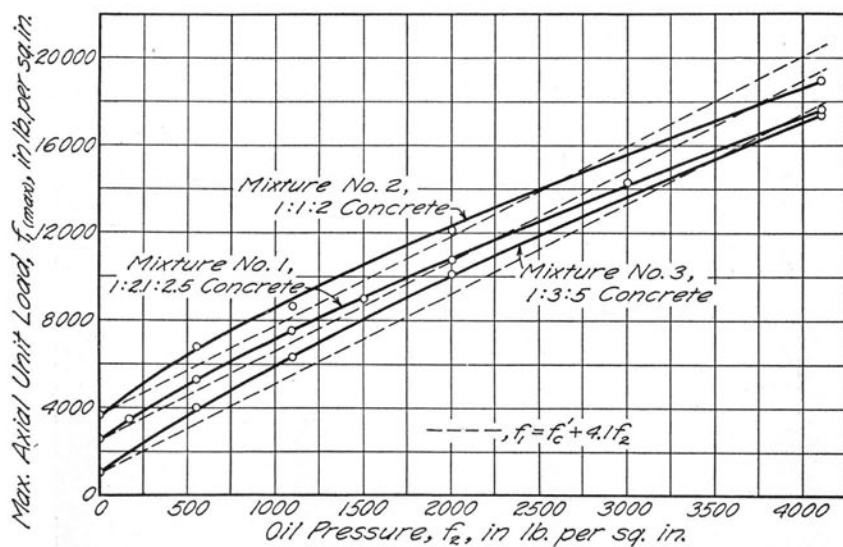


FIG. 22. MAXIMUM UNIT LOADS AND LATERAL OIL PRESSURES, SERIES 3A

TABLE 8
SUMMARY OF RESULTS OF TESTS OF SERIES 3A

Mixture	Lateral Oil Pressure, f_2 lb. per sq. in.	No. of Specimens	Maximum Axial Unit Load, f_1 lb. per sq. in.	Mean Deviation from Average Strength of Group		Average Initial Slope of Load-Deformation Curves lb. per sq. in.	Stress-Difference $f_1 - f_2$ lb. per sq. in.
				lb. per sq. in.	per cent		
No. 1 1:2:1:2.5	0	8*	2 575	305	11.8	3 420 000	2 575
	180	2	3 420	175	5.1	1 220 000	3 240
	550	3	5 240	255	4.8	1 330 000	4 690
	780	3	6 440	435	6.8	1 500 000	5 660
	1090	4	7 560†	130	1.7	1 360 000	6 470
	1510	4	8 930	145	1.6	1 230 000	7 420
	2010	4	10 800	375	3.5	1 250 000	8 790
	3010	4	14 260†	225	1.6	1 190 000	11 250
No. 2 1:1:2	0	4*	3 660	130	3.6	4 010 000	3 660
	550	4	6 770	70	1.0	1 700 000	6 220
	1090	4	8 620	240	2.8	2 170 000	7 530
	2010	4	12 150	300	2.5	2 000 000	10 140
	4090	4	19 160	290	1.5	1 900 000	15 070
No. 3 1:3:5	0	4*	1 050	30	2.9	2 550 000	1 050
	550	3	4 020†	145	3.6	950 000	3 470
	1090	4	6 260	70	1.1	660 000	5 170
	2010	3	10 175†	85	0.8	325 000	8 165
	4090	4	17 440‡	240	1.4	450 000	13 350

*Representing all batches of concrete from which triaxial test specimens were made.

†Three specimens included in average.

‡Two specimens included in average.

§Ultimate load not reached.

having passed the maximum when the tests were discontinued. Table 8 contains a summary of average results for all of the tests including those in simple compression. The average unit loads given in the table have been plotted against the corresponding oil pressures in Fig. 22. It is seen that the oil pressure had a marked effect in raising the maximum load, and that there was a definite relation between the intensity of the lateral pressure and the maximum load carried. The points representing specimens of Mixture No. 1, for which the largest number of intensities of oil pressures were used, follow a well-defined curve. At low oil pressures the slope of the curve corresponds to an increase in strength of about 5.1 times the oil pressure applied. For oil pressures less than 1000 lb. per sq. in., the curve is not a straight line; the rate of increase in maximum load with increase in oil pressure gradually decreases. For all pressures above 1000 lb. per sq. in., the curve approaches a straight line in which the increase in maximum load is about 3.5 times the increase in oil pressure. For Mixtures Nos. 2 and 3, fewer points were available for locating the curves.

However, the points fall closely upon curves having the same character as that drawn for Mixture No. 1. For pressures more than 1000 lb. per sq. in. the slope of the curve for Mixture No. 2 is the same as that for Mixture No. 1, while that for Mixture No. 3 corresponds to an increase in maximum load of 3.75 times the increase in oil pressure.

The similarity of the curves indicates that the added strength produced by a given lateral pressure is nearly constant, regardless of the mixture. It seems reasonable that the curves for the three mixtures should be similar, with ordinates differing only by the differences in strength of the specimens tested in simple compression. It may be noted that these curves agree fairly well with the results of the tests of spirally reinforced concrete of Series 1, in which the increase in the ultimate strength produced by the lateral pressure developed by the spiral reinforcement was equal to 4.1 times the lateral pressure. In other words, the maximum unit load carried may be considered as made up of the strength of the concrete in simple compression plus an added strength equal to 4.1 times the lateral pressure developed at the time of failure. While the solid curves of Fig. 22 vary slightly from the broken lines representing the foregoing relation, a reasonable conclusion of fundamental importance is that the law of resistance is essentially the same for concrete restrained by lateral oil pressure, in which the lateral bulging of the concrete is only incidental, as for concrete spirally reinforced, wherein the needed lateral resistance is developed only when a definite bulging or lateral swelling of the concrete has occurred.

26. *Variations in Strength of Companion Specimens.*—Table 8 contains a column giving the mean deviation from the average strength for each of the groups of companion specimens tested at the same intensity of oil pressure. Only the specimens for which values were included in the averages (see Table 7) have been considered in computing these deviations. The table shows that the deviations from the average are apparently independent of the intensity of oil pressure. The variations differ considerably for the different groups, but not in a consistent way. This fact, together with the similarity of the curves for the three different mixtures mentioned in the preceding paragraph, indicates that the added strength produced by the lateral pressure is not appreciably dependent upon the quality of the concrete. The differences existing between specimens, whether intentional variations in mixture or unavoidable differences in strength of speci-

mens of the same mixture, are not magnified when the strength of each specimen is raised by the presence of lateral pressure. These observations indicate that the increment of strength produced by the lateral pressure is a particularly reliable part of the total strength. It has been noted that the specimens in these tests were all made from the same lots of cement, sand, and gravel. It is probable that the added strength due to the lateral pressure is more dependent upon the character of the materials used than on the proportions of the mixture.

27. Comparison of Slopes of Load—Deformation Curves.—The slopes of the straight lines drawn as tangents to the load—deformation curves in Figs. 17 and 18 are recorded in Tables 7 and 8. A striking difference is seen between the average initial modulus of elasticity in the specimens tested in simple compression, and the slopes of the initial tangents for the specimens tested in three-dimensional compression, the latter being consistently only from one-fifth to one-half of the former.

The relations between loads and deformations given in Section 7 show the rate of deformation due to increasing axial load on an elastic material to be unaffected by the presence of a constant lateral pressure. It seems impossible to ascribe the difference in slopes to any action of the apparatus, since corrections were made for all probable errors in manipulation and observation. In the tests of Series 2, a difference in the drying out of the two types of specimens before testing was considered. In Series 3A the specimens that were surface treated with plaster of Paris were not covered with brass envelopes, and most of the water from the plaster had dried off when they were tested. Differences in moisture should affect the strength as well as the stiffness, and there is no indication of differences in strength properties between the specimens tested in simple and in three-dimensional compression. It does not seem possible that any difference in drying conditions could cause the large differences in stiffness indicated; rather it seems that the differences in slope must be the result of regular features of the deformations of the specimens tested at different amounts of oil pressure. In the first place, the deformations of the apparatus described in Section 22 at low excess vertical loads were so irregular that no accurate readings of the deformation of the specimen were obtained in the three-dimensional tests until an excess load of 1000 to 1500 lb. per sq. in. had been applied. The tangents to the load—deformation curves in Figs. 17 and 18 denoted as initial

tangents are therefore tangents to the curves at excess loads of 1000 to 1500 lb. per sq. in. Further, by analogy with the action of spirally reinforced columns, in which a compacting of the material in the "spiral range" is often very prominent, it seems probable that during the application of the oil pressure before strain measurements were taken some inelastic deformation of the specimens had already started, and that such deformation became more prominent the higher the oil pressure applied.

It is apparent that the two factors last mentioned would both have the effect of lowering the slope of the initial tangents of the load—deformation curves obtained in the three-dimensional tests. The fact that a certain amount of excess vertical load had to be applied before good measurements of deformations could be obtained would have a larger effect at the low oil pressures than at the higher ones, while inelastic volume changes would be more apparent in the tests conducted at high oil pressures. It seems reasonable to conclude that the two conditions may have acted together to produce the low values of the initial slopes observed.

28. *Characteristics of Deformation Phenomena.*—Figure 23 gives average load—deformation curves for specimens of the three concrete mixtures tested in triaxial compression. Each curve represents the average of three or four tests made with the same intensity of oil pressure. These averages were derived from individual curves such as those of Figs. 17 and 18, as follows: Horizontal lines were drawn in these figures at vertical unit loads equal to the intensity of the oil pressure, and the initial tangents to the curves were projected to intersect these lines. From these intersection points as origins, deformations and corresponding unit loads were read off from all curves, and the unit loads corresponding to definite increments of deformation were averaged for each group of specimens. The difficulty in representing the averages thus determined in one common diagram consists in locating the origin of the several curves due to the fact that considerable deformation had taken place before initial measurements were taken. It has been assumed that the initial tangents always intersected the horizontal lines representing the oil pressures at points on a straight line having the slope $\frac{E}{1-2\mu}$. The location of the curves may be slightly in error; if so, it is probable that the curves for the higher oil pressure have been placed somewhat too far to the left and those for the lower oil pressures too far to the right.

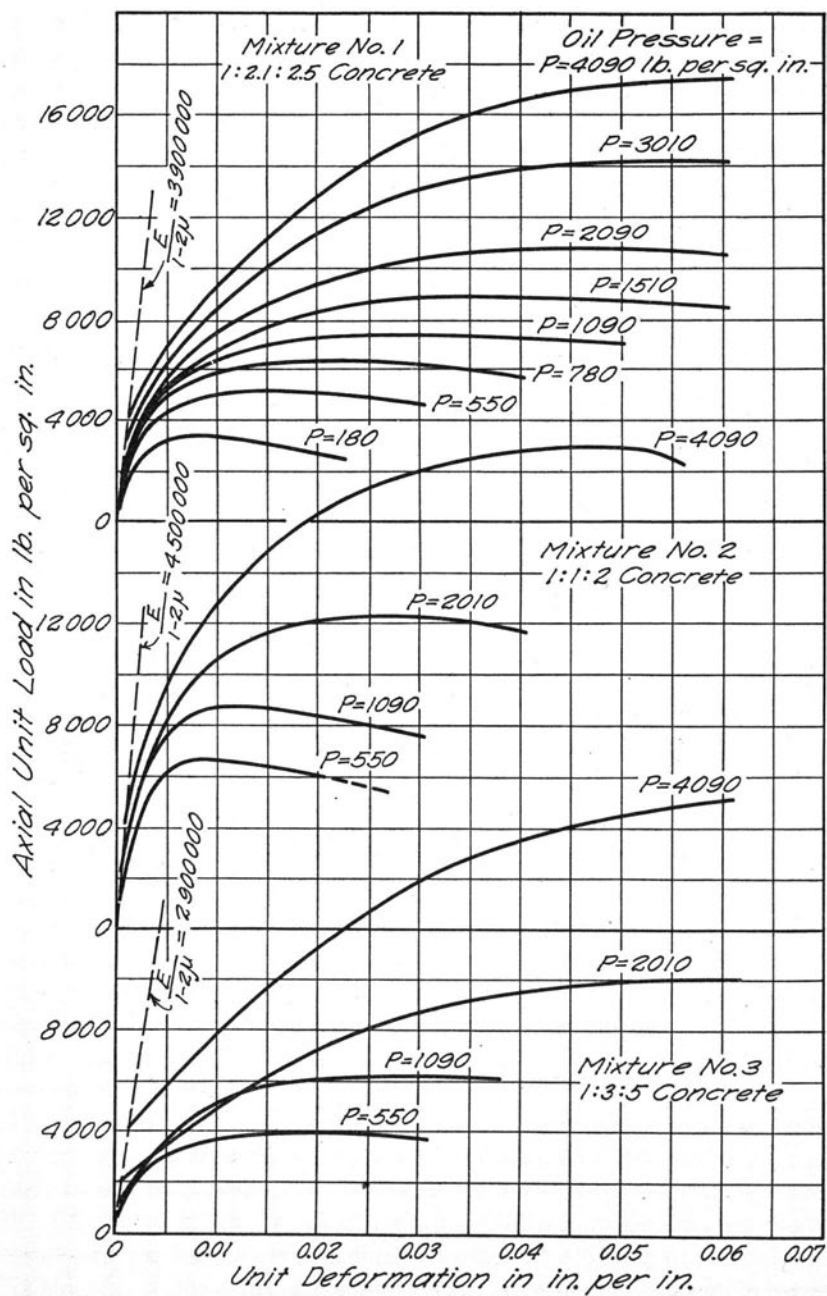


FIG. 23. AVERAGE LOAD-DEFORMATION CURVES, SERIES 3A

The broken lines in Fig. 23 represent the stress-strain relations according to which elastic deformations would take place during the application of a uniform compression on all sides of the specimen. The value of the modulus of elasticity was taken as slightly less than the average initial value obtained from the simple compression tests, and the value of Poisson's ratio was taken as 0.11. The average curves were then started directly from these broken lines at the vertical unit load corresponding to the respective intensities of oil pressure. No claim for great accuracy is made for this manner of locating the curves; presumably the beginning of the curves would not be quite as shown in the diagram, since the true curves would deviate from the broken lines at low loads.

Figure 23 shows the general trend of the deformations measured in these tests, and their variations with the mixture and with the oil pressure. It is seen that as the oil pressure increased the deformation at maximum load also increased very greatly. Thus, while the axial deformation at maximum load for specimens in simple compression was about 0.0015 in. per in., the corresponding axial deformation for the condition of 4090 lb. per sq. in. lateral pressure in Fig. 23 is more than 0.06 in. per in. It may also be noted that the shape of the various curves of Fig. 23 is quite similar at the beginning, the change in rate of deformation being about the same for all curves up to a deformation of 0.002 to 0.003 in. per in. A study of the curves leads to the conclusion that an inelastic reduction of volume, or compacting of the material, probably took place in all tests during the application of the oil pressure. This compacting was much more prominent in the case of the leanest mixture than for the other two mixtures.

29. Action of Material near Maximum Load.—In the tests of plain and spirally reinforced columns, (Series 1), it was found that in all cases the rate of lateral deformation showed a radical increase as the load approached the maximum, indicating that beyond a certain critical load the internal continuity of the material was rapidly being destroyed. In the tests of Series 2, the measurements of deformation were taken in a direction normal to the main applied stresses, and it was found possible to locate with some degree of certainty the critical point on the load—deformation curves corresponding to the load at which the process of breakdown of the material began to develop. It does not seem possible to locate a similar critical load for the specimens of Series 3A, since no critical point is apparent in the load—deformation curves measured parallel to the principal applied stresses. It seems probable, however, that a radical change in the volume of

the material did take place in the three-dimensional tests as the maximum load was approached, since the stress situations in these specimens were almost identical with those in spirally reinforced columns throughout the "spiral range" of the tests.

Whatever the internal nature of the action of the load may have been, it is known that the maximum load was reached while very large deformations were occurring. The specimen passed through a wide range of deformation with little change in the accompanying load.

30. *Condition of Specimens after Test.*—It is seen from the views of the specimens after test (see Fig. 21) that most of them show no superficial signs of injury due to the large stresses and deformations to which they had been subjected. Measurements taken before and after the tests show that very large permanent deformations were produced, and large variations in volume evidently took place. Some specimens tested at low oil pressures showed a net increase in volume, with indications that such increase took place in the later stages of the test. In the specimens tested at the higher oil pressures, however, such increase in volume was not great enough to overcome the decrease in volume produced in the earlier stages of the test; the result is a considerable net decrease of volume for most of the specimens tested at high oil pressures. Few of the specimens were found broken when they were removed from the pressure chamber. One specimen, No. 114, for which the lateral oil pressure was small, sheared off along a plane inclined less than 45 deg. to the axis. A number of the specimens showed cracks, most of which were inclined, on the surface, and irregularities in shape, thus indicating that a local displacement of the material, such as a sliding along inclined planes, had begun. A pronounced lateral bulging was also noted in many of the specimens, particularly in those subjected to high end loads and oil pressures. The views of the specimens can be identified by reference to Table 7.

V. CONCRETE IN TRIAXIAL COMPRESSION, SERIES 3B

31. *Outline of Series 3B.*—Series 3B included the tests of forty-eight 4 by 22-in. concrete cylinders in three-dimensional compression, the two principal stresses normal to the axis of the cylinders being larger than the axial stress. The pressure chamber was designed for a safe working pressure of 7000 lb. per sq. in., and this greatly limited the range of loading in the tests of this series. Furthermore, the high oil pressures necessary in these tests were a much more

severe tax on the surface covering of the specimens and on the packing of the joints than in previous tests. Originally, the scope of this series was limited to tests of concrete of one mixture which was the same as Mixture E of Series 2, and Mixture No. 1 of Series 3A. The only variable used was the load acting on the end of the specimen during the tests. Three intensities of end load were used, 500, 1000, and 1500 lb. per sq. in.; and four companion specimens were tested at each load. Later the tests were repeated with specimens made of the three mixtures of concrete used in Series 3A.

32. *Making and Preparation of Specimens.*—The making of the specimens was similar to that of specimens of Series 2 and 3A. The original tests, Group 1 of the series, were made on 4 by 22-in. cylinders with 4 by 8-in. companion control cylinders. In the tests of Group 2, made a year later, 4 by 8-in. cylinders were used for both the three-dimensional and the simple compression tests. All cylinders were capped with neat cement. Forms were removed 24 hours after pouring and the specimens were stored in the moist room until two days before testing. The preparation of the 4 by 22-in. specimens of Group 1 for testing was similar to the preparation of the specimens of Series 2. The cylinders were surface-treated with Lumnite cement mortar, followed by a coat of plaster of Paris, and finally covered with a double covering of sheet brass. Diameter measurements similar to those of Series 2 were taken before and after tests by use of a screw micrometer. The preparation of the 4 by 8-in. specimens of Group 2 was the same as that of the specimens of Series 3A (see Section 20). Care was taken to make the ends of the specimens perpendicular to the axis. The specimen was fastened by tape at the ends to two steel plungers and protected from the oil by rubber tubing.

33. *Arrangement of Testing Apparatus.*—The arrangement in the pressure chamber of the two types of specimen used is shown in Fig. 24. In applying axial load to the specimen two methods were used. In the first eight tests (Specimens Nos. E11 to E18) the axial load was applied by the 300 000-lb. testing machine and weighed by means of sensitive 10 000-lb. scales. Two loads exceeding 10 000 pounds were weighed directly on the testing machine. In the remaining 40 tests the vertical axial load was applied as a dead load by means of proving levers, as shown in Fig. 25. The pressure chamber was supported in a vertical position on the table of the testing machine, and increments of load were produced by adding weights to the scale pans suspended from the ends of the proving levers.

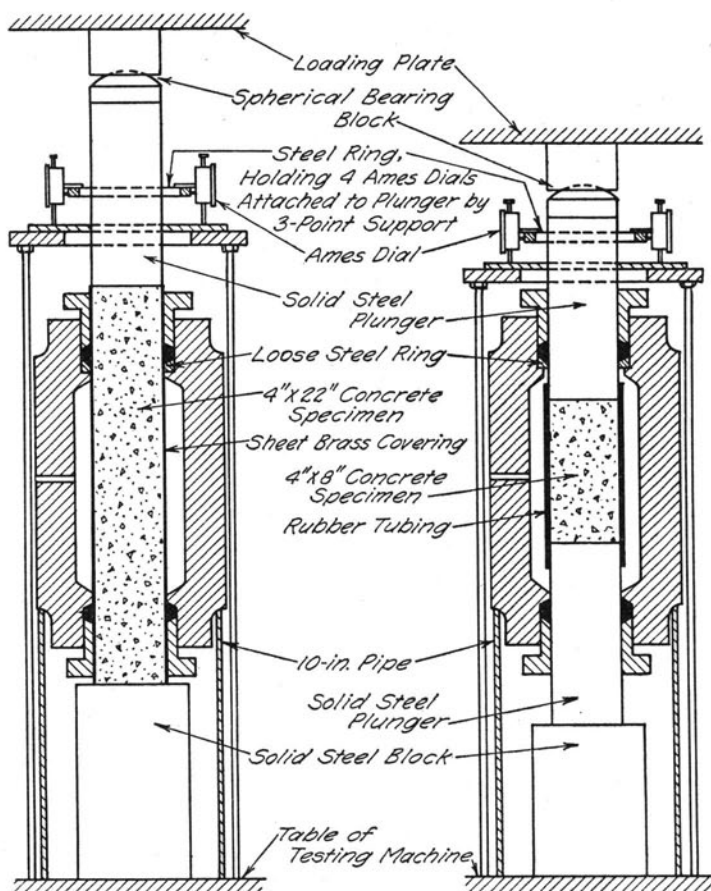


FIG. 24. ARRANGEMENT OF SPECIMENS IN PRESSURE CHAMBER, SERIES 3B

The arrangement for measuring the axial deformation of the specimen is shown in Fig. 24. The dial ring used in series 3A was placed on the steel plunger above the specimen. Due to the possibility that the pressure chamber might be lifted from its support as the specimen elongated, the movement of the dial ring was measured relative to a heavy annular steel plate supported by four $\frac{3}{4}$ -in. steel rods standing directly on the 10 000-lb. scales or on the table of the testing machine. The apparatus was found to be very satisfactory in regard to accuracy of measurement.

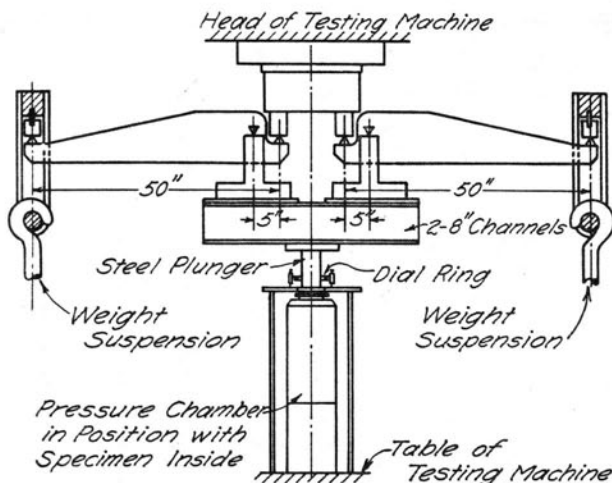


FIG. 25. LOADING APPARATUS USED IN SERIES 3B

In performing a test the end load, which was to be held constant throughout the test, was first applied. In the 1:3:5 specimens of Group 2 it was necessary to apply some oil pressure before the higher end loads could safely be applied. Zero readings of the micrometer dials for measuring deformations were taken, the oil pressure was applied in increments of 300 to 500 lb. per sq. in., and corresponding dial readings were made. With vertical load applied by the testing machine, the deformation of the specimen made frequent adjustment of the load necessary. In this regard the proving levers were more satisfactory, since the dead weight was independent of the deformation of the specimen. After the test, measurements of diameters and photographs of the specimen were obtained.

34. *Phenomena of Tests.*—During the early stages of loading the concrete cylinder elongated very slowly with the progressive application of oil pressure. Appreciable deformation of the specimen, indicated by an increase in end load in the tests by the first method, was generally noted at oil pressures of from 1500 to 2000 lb. per sq. in. The rate of elongation of the specimen gradually increased, particularly with specimens subjected to a low end load. At the higher oil pressures, deformation became rapid and a considerable creeping, or continued elongation under a constant oil pressure, was noted. This creeping made it difficult to maintain a constant end pressure by means of the testing machine, and led to the use of the dead weight applied through proving levers.

Of the twelve specimens of Group 1, only four were broken by the oil pressure, two at each of the lower intensities of axial load. The fracture in each case was on a plane normal to the axis of the cylinder. The specimen broke with a heavy shock and it was evident that the high oil pressure had come to act very suddenly on the cross-section of the specimen at fracture. Examination of two of these cylinders, E11 and E13, after failure showed a hole in the brass covering at the section of fracture which may have produced a premature failure of the specimen. The other two, E14 and E20, carried high loads and gave evidence of general and extensive breakdown of the concrete at the section of failure. The remaining eight specimens of the group, including the four subjected to the highest axial load, could not be broken. After these specimens had reached the point where a large plastic deformation or "creeping" had developed, the oil pressure was lost through the failure of a packing ring. This was due to two causes, the high intensity of pressure and the reduction of the diameter of the specimen.

In the tests of Group 2 much of the trouble of the previous tests was avoided by the use of steel plungers within the packing rings and by protecting the 4 by 8-in. specimens against penetration of oil by means of rubber tubing instead of the brass envelope. There was considerable irregularity in the results of all tests in which oil pressures in excess of 6000 lb. per sq. in. were employed. The specimens of 1:2.1:2.5 concrete with an axial load of 1505 lb. per sq. in. and those of 1:1:2 concrete with axial loads of 1030 and 1505 lb. per sq. in. could not be broken, even though oil pressures as great as 7300 lb. per sq. in. were applied in many cases. The tests of the 1:3:5 concrete at axial loads of 525 and 1040 lb. per sq. in. gave very consistent results, comparable in uniformity with the results of Series 3A. The other three groups in which failures occurred gave less satisfactory results, one or two specimens of each group being accidentally broken, or giving evidence of premature failure.

35. Results of Tests.—The principal results of the tests are given in Table 9 and in the typical pressure—elongation curves of Fig. 26. For the 4 by 22-in. cylinders a gage length of 15 in. was assumed in computing unit deformations; for the 4 by 8-in. ones the full length of the cylinder was used. Figure 27 gives typical curves for 4 by 8-in. cylinders tested in simple compression.

Views of specimens of Group 1 after test before and after removal of the brass covering are shown in Fig. 28. The figure shows a hole

TABLE 9
PRINCIPAL RESULTS OF TESTS OF SERIES 3B

Spec. No.	Max. Oil Pressure f_2 lb. per sq. in.	Unit End Load, f_1 lb. per sq. in.		Strength of Concrete f'_c lb. per sq. in.	Spec. No.	Max. Oil Pressure f_2 lb. per sq. in.	Unit End Load, f_1 lb. per sq. in.		Strength of Concrete f'_c lb. per sq. in.	Spec. No.	Max. Oil Pressure f_2 lb. per sq. in.	Unit End Load, f_1 lb. per sq. in.		Strength of Concrete f'_c lb. per sq. in.
		Applied	Corrected				Applied	Corrected				Applied	Corrected	
Group 1. 1:2.1:2.5 Concrete 4 by 22-in. cyls.														
E11	3210	495	610	2150	E12	5675†	825	950	2095	E16	5850*	1495	1635	2105
E14	5320	495	610	2210	E13	4430	825	950	2420	E18	5500*	1445	1585	1855
E15	4430†	495	610	1870	E17	500†	825	950	2160	E21	5500*	1465	1605	1860
E19	3740†	470	585	E20	6910	800	925	1485	E22	5150*	1485	1625	1530
Av.		490	605	2080	Av.		820	945	2160	Av.		1475	1615	2090
Group 2. 1:3:5 Concrete 4 by 8-in. cyls.														
31	3900	525	660	835	32	5325	1070	1245	730	33	3725**	1495	1630**	1030**
34	3725	525	660	805	35	5500	1070	1245	965	36	7175	1495	1700	1250
37	3725	525	660	1110	38	5325	1030	1205	1125	42	6400	1495	1700	1200
40	3975	525	660	1360	41	5500	990	1165	950	Av.	6790	1495	1700	1225
Av.	3830	525	660	1030	Av.		1040	1215						
Group 2. 1:2.1:2.5 Concrete 4 by 8-in. cyls.														
51	4675	525	685	1550	52	5125**	1070	1270	2130	53	5525*	1540	1740	2090
54	4950	525	685	1835	55	5125**	1070	1270	56	7300*	1495	1700	1960	
57	5125	525	685	2100	58	6925*	990	1190	2350	59	6575*	1495	1700	2450
60	4775	525	685	2430	61	6825	990	1190	2460	62	7300*	1495	1700	2300
Av.	4880	525	685	1980	Av.		1015	1215	2405	Av.	6675	1505	1710	2225
Group 2. 1:1:2 Concrete 4 by 8-in. cyls.														
71	5125	525	725	3250	72	5125*	1070	1260	3460	73	7300*	1540	1740	3800
74	6750	525	725	3750	75	7300*	990	1200	3920	76	7300*	1495	1700	3240
77	6750	525	725	3750	78	7300*	990	1200	3500	79	7300*	1495	1700	3350
80	6050	525	725	3570	81	6210	990	1200	3770	82	7300*	1495	1700	3560
Av.	5975	525	725	3570	Av.		1030	1230	3480	Av.	7300	1505	1710	3490

*Packing gasket blew out. †Not sufficient to cause failure. ‡Max. load not observed. ¶Broken by initial end load. §Rubber tubing punctured. **Omitted from average.

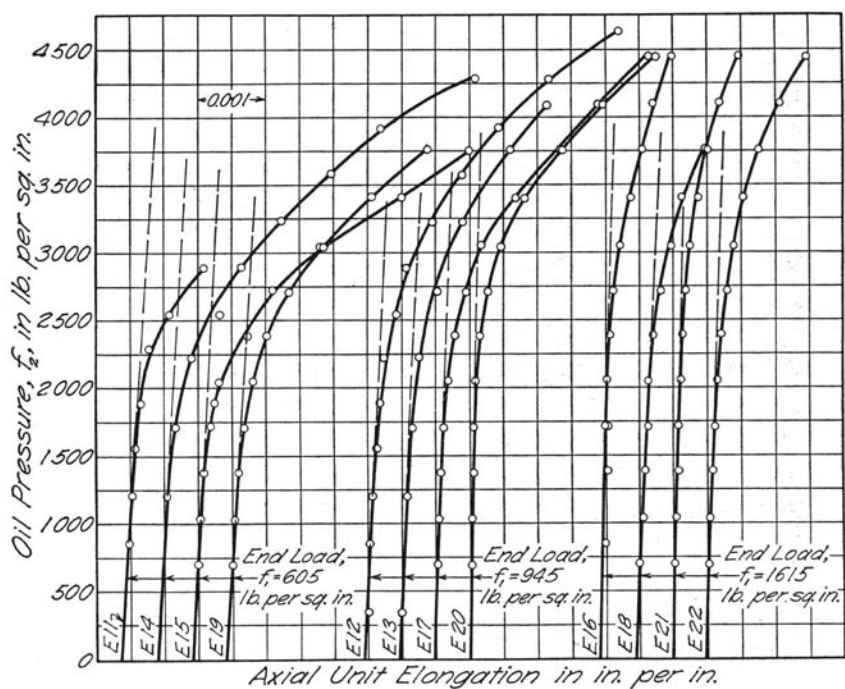


FIG. 26. PRESSURE-ELONGATION CURVES AT LOW LOADS FOR SPECIMENS OF SERIES 3B

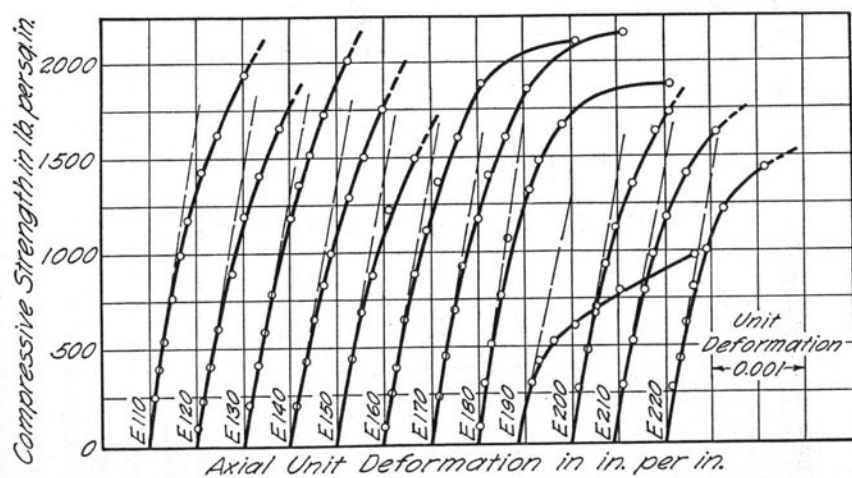


FIG. 27. LOAD-DEFORMATION CURVES FOR SPECIMENS IN SIMPLE COMPRESSION, SERIES 3B

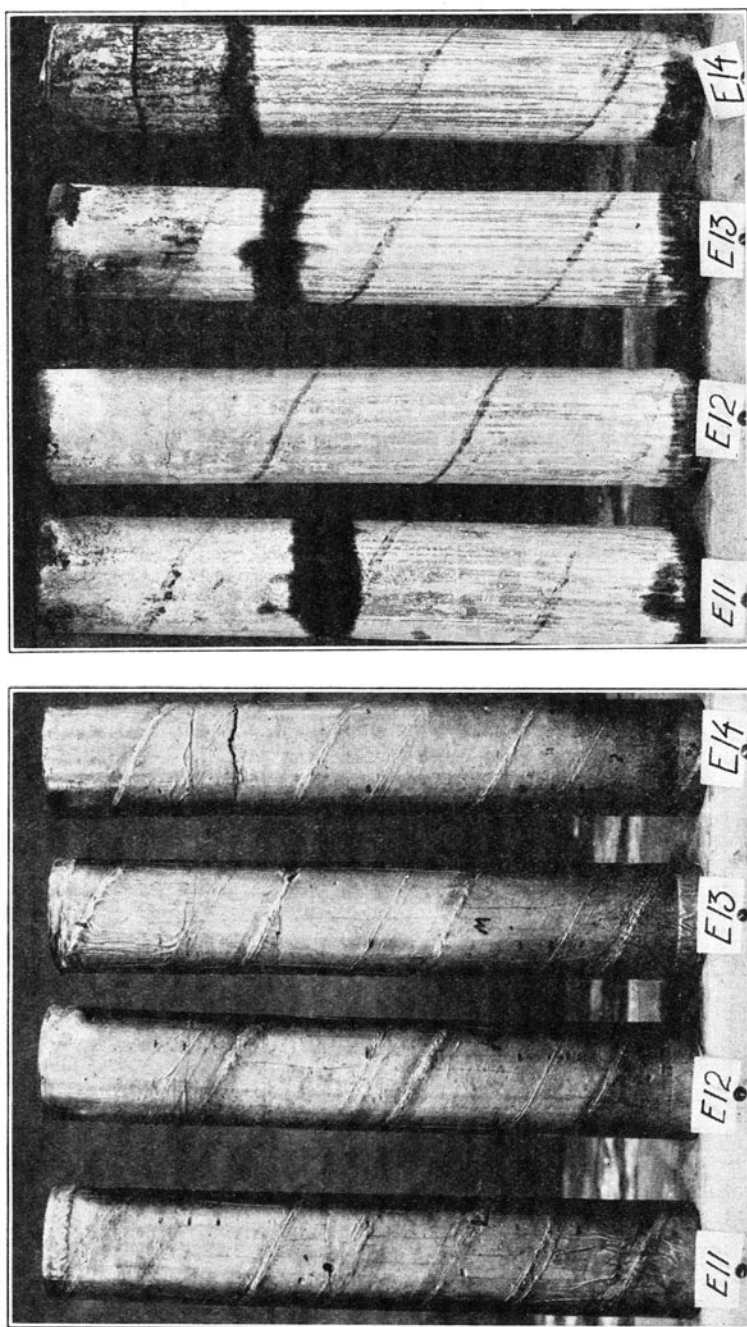


FIG. 28. SPECIMENS OF SERIES 3B AFTER TEST, BEFORE AND AFTER REMOVAL OF BRASS COVERING

in the brass covering at the section of fracture of both Specimens E11 and E13, indicating that failure was caused by penetration of oil. Specimen E14 shows no surface cavities, but the depression produced on the specimen by the packing rings indicates a large deformation of the specimen before failure. An interesting feature of Specimen E17, not shown in the figure, was a surface depression about $\frac{1}{4}$ in. deep and more than $\frac{1}{2}$ in. in diameter, probably produced by a cavity beneath the surface of the concrete. The depression was evidently formed by plastic flow of the concrete under pressure, neither the surface of the concrete nor the brass covering being broken, though the latter was pressed tightly into the depression.

The specimens of Group 2 were not subjected to the pressure of the packing rings and the rubber covering conformed well to surface irregularities of the specimens, though there remained a possibility of premature failure due to rupture of the rubber envelope. In all of the tests the large inelastic reduction in volume during testing obscured the relations between loads and deformations, and made it difficult to establish a criterion as to the critical load carried. Load—deformation relations were also rendered uncertain by the necessity in some of the tests of applying some lateral pressure before the constant end load could safely be applied. The method of testing used for Group 1, with the uncertainty as to penetration of oil, the high deformation within the packing rings, and the presence of secondary stresses due to loading only a part of the length of the specimen, may be considered a failure, except as a means of evolving a better method. The method finally developed appears to be fairly satisfactory, though it is hoped that means of applying higher pressures may be found for future tests.

36. *Deformations at Low Loads.*—It is believed that the tests of Series 3B gave reliable information on the deformation of the specimens at low loads, even though there is some doubt as to the correctness of the ultimate loads attained. In testing, the end load was first applied and remained constant throughout a test. The accompanying deformations of the loading apparatus should not influence the subsequent deformation readings, except possibly in the interval between the zero readings and the first reading after application of oil pressure, a period in which a reversal in the direction of axial deformation occurred. Table 10 gives a record of the initial slopes of the pressure—elongation curves. With the end load constant throughout the tests, these slopes should be equal to the ratio $\frac{E}{2\mu}$ and should be inde-

TABLE 10
CRITICAL PRESSURES FOR SPECIMENS OF GROUP 1, SERIES 3B

Specimen No.	Axial Unit Load lb. per sq. in.			Critical Oil Pressure* lb. per sq. in.	Av. Initial Slope of Pressure-Elongation Curves lb. per sq. in.
	Applied by Proving Levers	From Friction and Brass	Total		
E11, E14, E15, E19 E12, E13, E17, E20	490	115	605	2500	8 560 000
E16, E21, E22	820	125	945	3080	11 750 000
	1475	140	1615	3980	16 600 000

*Pressure at which the slope of pressure-elongation curve was one-tenth the initial slope.

pendent of the end pressure if the materials were acting elastically. It is seen that the latter condition did not obtain. Apparently there was some inelastic action at pressures even lower than that at which the first readings were taken, usually about 600 lb. per sq. in. The initial slopes of the curves were raised considerably by an increase in the end load; this seems direct evidence that an inelastic compacting of the material took place at low pressures and that it was more prominent with the higher end loads. It is seen that a compacting or a reduction in volume of the material greater than that corresponding to the elastic deformation under stress would in this case reduce the axial elongation and produce a steeper slope of the pressure—elongation curve.

37. *Effect of End Load on Maximum Pressures.*—From the results of Series 2 it appeared that breakdown of the material began near the point at which the slope of the pressure—elongation diagram was one-tenth of the initial slope. Assuming that a similar condition might obtain for the specimens of Group 1 of Series 3B, points were located on the pressure—elongation curves of Fig. 26, and values of the pressure at these points are given in Table 10, together with values of the effective end load, which includes the end restraint produced by friction and brass covering. There is little in the appearance of the curves of Fig. 26 to indicate that these points are "critical" points for Series 3B, so that they have been used only for the purpose of correcting the critical loads in Series 2 for the influence of end restraint. The values of Table 10 indicate that at the so-called critical point the presence of end load produced an increase in the oil pressure equal to about 1.6 times the end pressure.

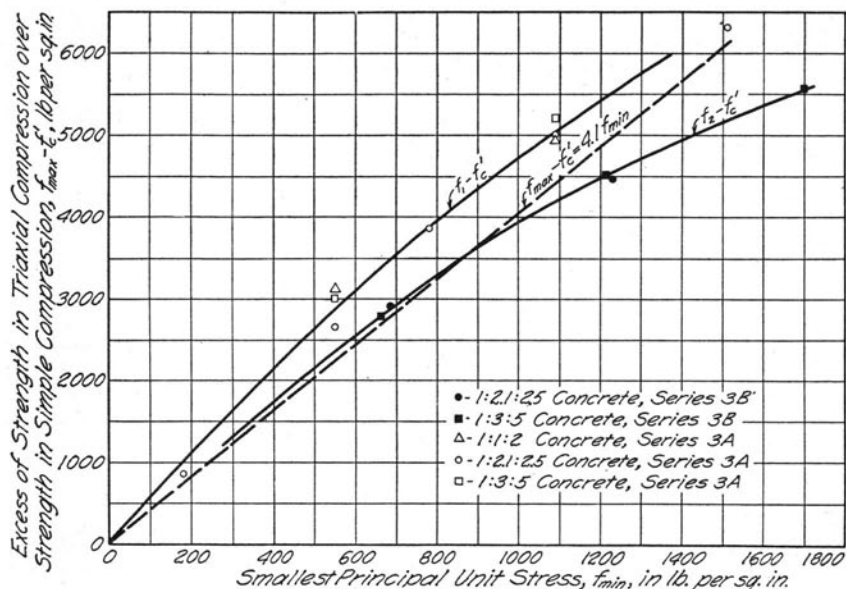


FIG. 29. MAXIMUM PRESSURES WITH VARIOUS END LOADS, SERIES 3B

The relation between the maximum oil pressure and the end load in Series 3B differs radically from the relation noted above for critical loads. Using the results of Group 2, Table 9, which give values of the oil pressure at failure, the relation between end load and maximum oil pressure has been plotted in Fig. 29. For comparison, the curve $f_{\max} - f_c' = 4.1 f_{\min}$ and the curves of Fig. 22 representing the results of Series 3A have also been plotted in the figure. The end loads plotted in Fig. 29 include frictional restraint calculated from the data of Fig. 15. It is seen that at the value of the smallest principal unit stress f_{\min} equal to about 500 lb. per sq. in. there is a distinct difference between the results of Series 3A and 3B, and a still greater difference at a value of f_{\min} equal to 1000 lb. per sq. in. With f_{\min} equal to 1500 lb. per sq. in. the only value obtainable for Series 3B is the result of two tests of 1:3:5 concrete, and it does not appear very reliable. This value is considerably less than the corresponding value from Series 3A.

From consideration of Fig. 29 it may be said that for tests in which the applied oil pressures did not exceed about 6000 lb. per sq. in. the strengths of the specimens of Series 3B were 15 to 20 per cent less than those of Series 3A. Much of this difference is due to the correction for friction that was applied to the results of series 3B, and

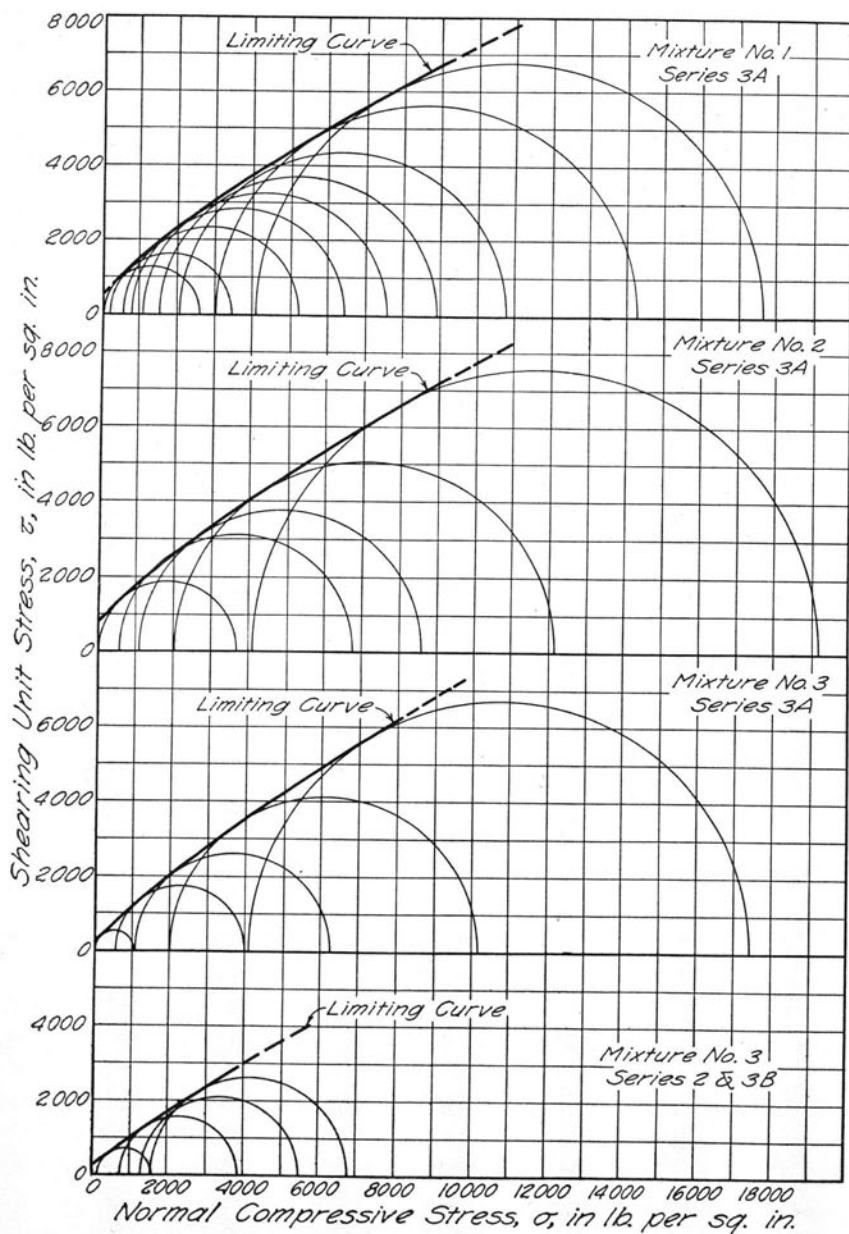


FIG. 30. "MOHR'S CIRCLE" DIAGRAMS FOR CONCRETE IN TRIAXIAL COMPRESSION

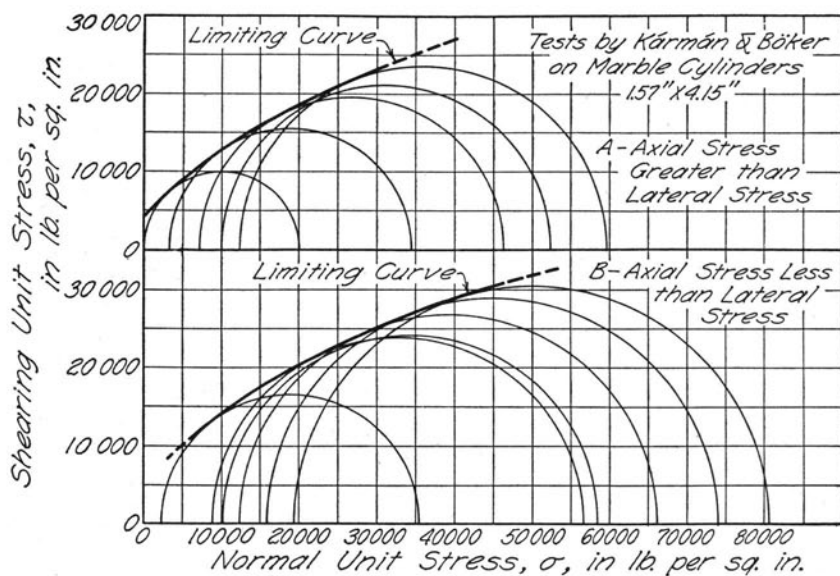


FIG. 31. "MOHR'S CIRCLE" DIAGRAMS FOR MARBLE IN TRIAXIAL COMPRESSION

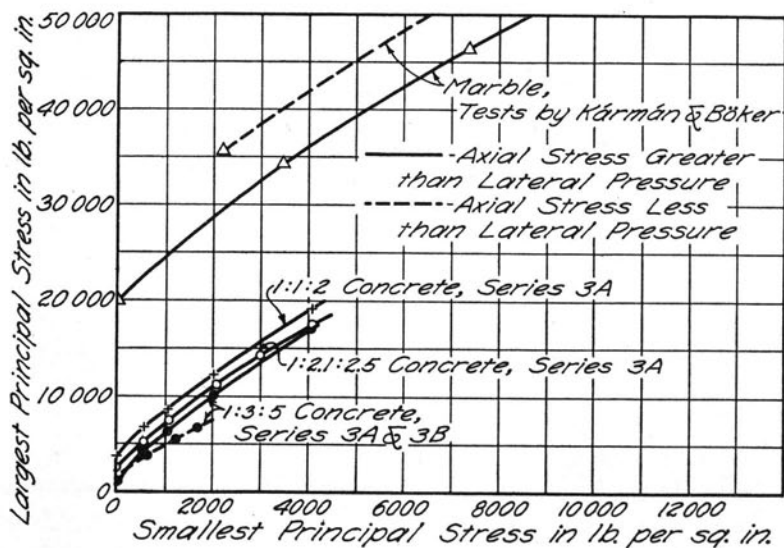


FIG. 32. MAXIMUM AND MINIMUM PRINCIPAL STRESSES IN CONCRETE AND MARBLE

hence represents rather indirect evidence regarding essential differences between the two kinds of tests. As will be noted later, the tests of Kármán and Böker and the analysis by Brandtzaeg indicate that the increment of strength due to the end pressure in Series 3B would be expected to be greater rather than less than that due to the oil pressure in Series 3A. In view of this disagreement, the limited number of tests of Series 3B, and the mechanical difficulties attending the use of the high oil pressures, it must be realized that these tests do not furnish conclusive evidence regarding the influence of the intermediate one of three principal stresses upon the failure of the material.

VI. GENERAL DISCUSSION

38. *Strength of Concrete in Triaxial Compression.*—Various combinations of the three principal stresses developed at the point of maximum load in the tests of Series 3A and a part of Series 2 and 3B are represented in Fig. 30 by "Mohr's circles," constructed as noted in Section 4. For comparison, similar curves are shown in Fig. 31 for the tests made on marble specimens by Kármán and Böker, previously mentioned. The first part of Fig. 31 shows curves for tests in which one principal stress was larger than the other two, corresponding to Series 3A, and the second part of the figure gives curves for the case corresponding to Series 3B. The results of Series 3A and 3B and of Kármán and Böker's tests of the strength of various materials in triaxial compression are summarized in the curves of Fig. 32, in which the greatest principal stress is plotted against the smallest principal stress. Figure 30 shows that when the results of each individual group of concrete tests are represented by Mohr's circles, they arrange themselves in such a way that they define limiting curves such as Mohr, on the basis of his analysis of the process of failure, concluded must exist for any given material. The fact that such limiting curves may be found to fit the results of a single set of tests with similar stress combinations in all tests does not prove the correctness of Mohr's theory. The regularity of these limiting curves merely proves that the strength of the material as measured by the largest compressive stress acting at the point of maximum load varies in a regular way with the magnitude of the smaller compressive stresses acting. To prove the correctness of the conception of failure underlying Mohr's theory, it would be necessary to show that this variation is the same regardless of the particular type of test, and that the intermediate principal stress had no influence. Figure 31 shows that Kármán's and

Böker's tests failed to substantiate Mohr's theory, since the limiting curve for the case where two of the principal stresses were larger than the third lies above the limiting curve for the other case. Figure 30 shows that in the tests of concrete reported here there was some difference between the limiting curves for the two cases; but the difference found was opposite in sense to that shown in Fig. 31. The limiting curves for the tests of Series 3A agree in their general trend with the corresponding curves from the tests on marble; in spite of the great difference in the character of the two materials they have essential characteristics in common. The tests of Series 2, which indicate that the strength of concrete in two dimensional compression is somewhat higher than that in simple compression, agree with the trend of Kármán's and Böker's tests on marble. The tests of Series 3B, while differing slightly from the limiting curve of Series 3A, are so limited in number, and the attendant uncertainties so evident, that it is felt that these tests are not sufficient to prove or disprove laws involving comparatively small differences in the limiting curves of the Mohr diagrams. The investigation should be continued with a larger number of tests, with apparatus capable of higher intensities of oil pressure, and with concrete made from aggregates of different characteristics, for which variations in internal friction might be found. In the absence of such tests it must be admitted that the present tests do not furnish conclusive evidence as to the correctness of Mohr's theory.

While the tests do not settle the scientific point at issue, they do show consistently that the strength of concrete is very materially raised by the presence of lateral compressive stresses. For the concrete used, the results of both Series 3A and 3B indicate an increase in strength of about 4.1 times the magnitude of the smallest lateral compression, thus agreeing also with the results of spirally reinforced concrete of the same quality tested in an accompanying series.

39. Condition of Material after Tests in Triaxial Compression.—As previously mentioned, most of the specimens tested in three-dimensional compression in Series 3A, and some in Series 3B, were still intact after the test, in spite of the large deformations to which they had been subjected. Similarly, material from the cores of the spirally reinforced members of Series 1, with the exception of those in which the spiral broke very suddenly, was apparently intact after tests, after having been shortened by several per cent of its length.

A number of the specimens of Series 3A were tested in simple compression some time after they had been tested in triaxial com-

TABLE 11
STRENGTH OF SPECIMENS OF SERIES 3A, RETESTED IN SIMPLE COMPRESSION

Spec. No.	Age in days at		Conditions at Original Test				Compressive Strength* when Re- tested lb. per sq. in.
	Original Test	Second Test	Maximum Load lb. per sq. in.	Oil Pressure lb. per sq. in.	Permanent Unit Deformation		
					Axial	Lateral	
113	28	80	3 240	180	0.0098	0.0079	2100
123	28	80	5 190	550	0.0235	0.0156	1470
124	28	99	4 910	550	0.0440	0.0205	1300
133	28	80	6 150	780	0.0317	0.0114	1740
142	28	148	7 650	1090	0.0348	0.0162	2105
153	28	80	8 710	1510	1540
163	28	80	10 650	2010	0.0532	0.0190	1680
164	28	99	10 550	2010	0.0611	0.0185	1460
173	28	80	14 100	3010	0.0526	0.0202	1375
183	28	80	17 650	4090	1550
184	28	99	17 600	4090	0.0712	0.0202	1430
Av.							1615
262	29	29	12 750	2010	0.0318	0.0146	2320
361	29	29	10 180	2010	0.0534	0.0198	770
342	28	28	6 320	1090	0.0356	0.0164	990
382	29	29	17 150	4090	0.0536	0.0202	770
384	28	28	24 600	6560	0.0780	0.0200	1000
Av.							880

*Based on original cross-section of cylinder; see Table 7 for data on original strengths of the concrete used.

pression. Table 11 contains results of these tests. The age of the specimens when retested varied from 80 to 148 days, whereas the age at the initial test was 28 days. However, since the specimens were stored in the dry air of the laboratory, there was probably little change in the strength between tests due to age. Referring to Table 11, there does not seem to be any consistent relation between the strength found in the second test and the maximum load or the oil pressure used in the first test. It may be noted however, that the tests of Series 3A were much alike for all specimens in that they were carried only to a point where it was certain that the maximum load had been passed. If it is considered that the decrease in load beyond the maximum for a specimen under constant lateral pressure is due to a slow destruction of the cohesive strength of the material, it will be realized that the tests of all specimens of Series 3A were discontinued at about the same point so far as this destruction of cohesion is concerned. Hence it might be expected that the strength at the re-

test would not bear a consistent relation to the stresses or deformations in the first test. Taking the average for each of the three mixtures of concrete, the compressive strengths found in the second test were, for Mixture No. 1, 1615 lb. per sq. in., for Mixture No. 2, 2320 lb. per sq. in. and for Mixture No. 3, 880 lb. per sq. in., or 63, 63, and 84 per cent, respectively, of the strengths in simple compression at the age of 28 days.

In some of the tests recorded in Table 11, deformations were measured. These deformations were quite different from those usually found for concrete. The rate of deformation was very high throughout the tests, corresponding to a modulus of elasticity of about 600 000 lb. per sq. in. for Mixture No. 1, and the deformations reached before failure were very large, more than twice as great as are usually found at ultimate load on concrete specimens.

40. *Volumetric Deformations in Concrete in Combined Compression.*—In the discussion of the results of the various series of tests, considerable attention has been given to the volumetric changes observed. In the tests of spirally reinforced columns, direct measurements showed that large inelastic reduction in volume took place, or that the material was compacted under high three-dimensional stresses. The tests of Series 3A and 3B did not give opportunity for direct measurements in three directions, from which changes in volume could be determined. However, the general trend of the deformations observed can be explained only by assuming that a large inelastic reduction in volume took place.

The probability that large inelastic deformations will accompany high stresses in three directions is of considerable importance in cases where stresses are computed on the basis of the assumed deformations, and where the increased strength of concrete in three-dimensional compression is utilized. The result of inelastic volume changes will always be an increase in the shortening of the material in the direction of the largest compression and a reduction of the expansion in the direction of the smallest stress. Where the magnitude of the smallest stress depends upon the deformation taking place in the direction of that stress, as in the spiral column, the value of Poisson's ratio becomes very important. The result of the tests emphasizes strongly that the high stresses that concrete can resist in three-dimensional compression will always be accompanied by very large deformations, even when the expansion of the material normal to the main compression is restrained or prevented.

41. *Relation of Test Results to Various Theories of Failure.*—Of the general theories of failure of materials mentioned in Section 4 only the maximum strain theory and some forms of the "planes of least resistance" theory have been seriously considered to apply to concrete. The two theories and their implications when applied to the failure of concrete in compression have been discussed in Section 4. The comparison of these theories with the observations made in the tests may be of value.

It was shown in Section 4 that, when applied in its original form, computing the strains due to combined compression from the equations of elasticity, and stipulating that the tensile strain must not exceed a fixed limit, Saint Venant's maximum tensile strain theory would indicate that the strength of a material in biaxial compression must be one-half of the strength in simple compression. The tests of Series 2 showed that the strength of concrete in biaxial compression was at least equal to its strength in simple compression, and probably considerably higher. The results of these tests show definitely that the maximum strain theory applied as stated cannot give direct criteria of failure for concrete. However, since concrete does not follow Hooke's law at stresses approaching failure, it is possible that the extension of the material in a direction normal to the largest compressive stress may still be a governing factor at failure, although this extension cannot be directly computed from the equations of elasticity. The tests of Series 2 and a set of simple compression tests on plain concrete gave opportunity for direct measurement of the extension of the material normal to the applied stresses throughout the tests, thus making possible a direct comparison of these deformations at failure in simple and in two-dimensional compression. It was found that the extension normal to the compressive stresses increased very rapidly before ultimate failure took place, and a certain critical point was determined where failure apparently began. The mixture used in the simple compression tests was that of Mixture E of Series 2. The lateral unit deformation of the simple compression specimens at the critical point was about 0.00017. The corresponding unit elongation in the tests of Series 2 was 0.0006. The difference is so large that it must be concluded that a fixed limiting value of the tensile strain is not the factor governing failure under all combinations of stress.

Coulomb's internal friction theory indicates that the strength of concrete in biaxial compression is equal to its strength in simple compression and that the strength in triaxial compression increases in proportion to the magnitude of the smallest principal stress, while the

magnitude of the intermediate principal stress has no effect upon failure. As previously stated, the tests of triaxial compression in this investigation give little direct information regarding the influence of the intermediate principal stress. The results of the tests of Series 3A are in fair agreement with Coulomb's theory in one respect: the relation found between the maximum load and the intensity of the oil pressure was very nearly a linear one. It was also found that the relation was quite similar for concrete of different mixtures. This may agree with an internal friction conception of the phenomena, since it is reasonable that the internal friction would depend more on the character of the various materials used in the mixture than on the proportions in which they were mixed.

The immediate results of the tests of Series 2 also agree with Coulomb's theory to a certain extent, since the specimens tested in biaxial compression withstood stresses at least as great as those in simple compression. Although in most cases the strength in biaxial compression was probably considerably greater than that in simple compression, this in itself is hardly enough to discredit the internal friction theory. It may be said that while the tests of Series 2 do not definitely disprove the theory, the indications are not in its favor.

Guest's maximum shear theory, considered as a special case of Coulomb's theory, is obviously inapplicable to concrete, as may be readily seen from Fig. 30.

Coulomb's theory contains more than a simple statement of a criterion of failure. The basis of the theory is a conception of failure as taking place through a sliding or shearing along continuous planes inclined to the direction of the main compressive stress. The simple compression tests gave opportunity for complete measurements of the deformation of the material throughout the loading; in several cases, these measurements were continued considerably beyond the ultimate load on the column. Since the theory is based on a definite type of deformation, it is evident that strain measurements must show such a deformation existing if the conception of failure is correct. The deformations actually measured showed consistently that the loading of the specimen at and beyond the point of maximum load was accompanied by a very rapid increase in the lateral deformation which was not paralleled by a similar increase in the longitudinal deformation. Such deformations are incompatible with a conception of failure occurring through a sliding on inclined planes or conical surfaces. A movement of diagonal slices of the material on each other must certainly affect the axial deformations as well as the lateral deformations

of the prism. If the inclination of the planes of least resistance agrees with Coulomb's theory (less than 45 deg. to the axis of the specimen), the effect of the sliding on the axial deformation must be even greater than on the lateral deformation.

It may be noted further that the deformations that characterized the passing of the maximum load in the simple compression tests indicated a splitting of the material rather than a sliding. While the other series of tests gave no opportunity for such complete measurements of the deformations as in Series 1, the general trend of the deformations measured also indicated the splitting failure, and hence the theory of a failure solely by sliding along continuous inclined planes does not seem to be a correct one for concrete.

Summarizing the results, it may be said that the "maximum tensile strain" theory gives criteria of strength under different conditions which are far from agreement with the results of these tests of concrete, while the observed deformations accompanying failure, indicating a failure through a splitting action, agree with a conception of the lateral expansion of the material as being the main source of danger. On the other hand, the "internal friction" theory, applied under the assumption of an isotropic material, gives criteria of strength which are not definitely proved to be incorrect, but the theory does imply a conception of the physical process of failure which is in marked disagreement with the deformations observed in the tests.

42. *A Conception of the Failure of Concrete in Compression.*—The foregoing tests provide a means of judging of the applicability to concrete of the analysis of failure of a non-isotropic material of a specific kind. This analysis, described in Section 6 and in the Appendix, retains certain features of the internal friction theory as applied to the deformation of the small non-isotropic elements of which the material is composed. It is assumed for the analysis that there is a limiting value of the shearing stress along the direction of weakness for each small element of the material, and beyond this stress a plastic sliding failure occurs; furthermore, during the development of plastic deformation all elements deform alike, so that the elastic elements carry a high proportion of the total load. The result of the analysis is that in the assumed material a progressive application of compressive stresses leads to the creation of lateral tensile stresses which eventually produce a splitting failure of the material. The analysis is more concerned with the plastic action leading up to failure than with the failure itself.

The properties of the assumed material of Section 6 do not agree in some respects with known properties of concrete, the principal dif-

ference being due mainly to certain assumed geometrical relations introduced in the analysis to facilitate mathematical procedure; however, such differences should not essentially affect the qualitative results of the analysis. As to the details of breakdown of the material, it can hardly be expected that all elements will deform alike, as assumed, after extensive plastic deformations have developed.

It is found by comparing the results of the analysis with the observations made in the various tests that qualitative agreement exists on several important points:

(1) The analysis indicates the existence of dangerously high lateral tensile stresses after extensive plastic action has developed. The tests indicated that failure started with the breaking of the continuity of the small parts of the material through a lateral bulging, and a splitting across surfaces parallel to the main compressive stress.

(2) Assuming values of elastic constants reasonable for concrete, stress-strain relations have been computed from the analysis which agree very well with the actual relations found in the tests, up to the point at which the lateral tension in some parts of the material becomes high.

(3) Assuming that the magnitude of the lateral tensile stress is the sole criterion of the stress situation causing the splitting failure to develop, the analysis indicates that the strength of the material in two-dimensional compression will be somewhat higher than that in simple compression. This agrees with the test results of Series 2.

(4) The strength in three-dimensional compression is found by the analysis to vary with the magnitude of the smallest principal stress in a way similar to that observed in the tests of Series 3A. Under stress conditions like those of Series 3B the analysis indicates an increase in strength with increase in lateral pressure somewhat more rapid than in Series 3A. This result agrees with the tests on marble by Kármán and Böker, but is in disagreement with the results of the tests of Series 3B. If the results of Series 3B are verified by further tests, agreement with the analysis may still be established by modifying the assumption that the splitting failure is dependent solely on the magnitude of the lateral tensile stress. This is an arbitrary assumption, although it is in agreement with indications of tests of marble cylinders in three-dimensional compression combined with torsion, to which reference is made in Section 5.

Following the reasoning of Section 6 some essential features of the breakdown of concrete in compression suggest themselves. It appears that the departure from elastic action begins as a plastic sliding failure along small inclined surfaces scattered throughout the ma-

terial. This sliding may be a bond failure between cement or mortar and a portion of the surface of a grain of aggregate, so that the plane of weakness may evidently be inclined in any arbitrary direction. It seems reasonable that the resistance to bond slip or plastic sliding may be partly of the nature of adhesion and partly an internal friction. With the spreading of this plastic sliding with increasing loads, tensile stresses are developed in those parts of the material still deforming elastically, and two distinct phenomena may follow:

(1) When the tensile stress in some parts of the material becomes too great, lateral splitting of the material will begin. Since the tensile strength varies in different parts, this action will extend through a range of loading, the splitting of the material finally becoming so extensive that the material cannot resist further load and the maximum load will be passed. The range through which this splitting failure will extend and the rapidity of the action will depend much on the strength properties of the material. Presumably this process of failure in concrete of very rich and strong mixtures may lead to sudden failure through an explosive splitting, the very small fractures accumulating almost instantaneously into continuous cracks throughout the material. A splitting action in concrete of a lean mixture may lead merely to a very slow crushing or crumbling failure, and in intermediate grades of concrete it should produce a condition between the two extremes.

(2) As the plastic action continues to spread to an increasing number of points, a stage will be reached in which the material as a whole approaches plasticity. Plastic sliding will proceed along a number of surfaces, so that the parts of the material in which no plastic action has occurred will be displaced relative to each other, and cannot further govern the deformation of the material as a whole. The process has been described as a disorganization of the material. As the material becomes more and more of a plastic nature, one direction of sliding motion may become prominent among the many directions along which sliding is proceeding, this one direction being dependent on the external stresses, and the external conditions of the deformation. It is probable that during this process of disorganization, certain directions of sliding motion may eventually become so much more prominent than all others that a condition may be approached which is similar to that assumed in the theories involving a plane of least resistance. If so, it seems reasonable that at this stage the relation of the external stresses on the material must be fairly similar to those defined by the internal friction theory in its original form.

The two results of the spreading of plasticity through the material, a splitting due to lateral tensile stresses, and a disorganization, may occur simultaneously and may influence each other. For instance, in a concrete of medium richness in simple compression the development of lateral tensile stress in small parts of the material will cause splitting failure of these parts to begin. Cracks will then develop in small parts of the material at a large number of points scattered throughout the cylinder; these cracks will gradually accumulate into continuous axial cracks throughout the material, which may become visible on the surface. However, as soon as tensile failure occurs in some parts of the material, this will cause a reduction in the normal compression on planes in the directions of weakness of other parts, and thereby will accelerate further spread of plastic action throughout the material. It is probable that the failure in lateral tension will be overtaken by the disorganization of the material, which will cause sliding failure to extend more or less continually through the material in directions approximately as given by the internal friction theory. Thus, a specimen which started to fail due to lateral tension may show a final failure by shearing, or a combination of shearing and splitting failure.

The process of breakdown of a concrete cylinder in two-dimensional compression under this conception of failure is essentially the same as that under simple compression with the difference that under this type of stress, since the frictional resistance against sliding motion acts on a larger number of surfaces than in the case of simple compression, the external stress required to start splitting is somewhat higher.

In three-dimensional compression, when the smallest principal stress is constant, a splitting failure also finally occurs. The permanent volumetric deformations found in the tests of Series 3A show that this splitting took place in many cases. However, the extent of plastic action necessary to develop tension in parts of the material when compression is acting from all sides, is higher than in the case of simple and two-dimensional compression; consequently, it must be assumed that disorganization of the material will play a greater part in the failure, and, as the magnitude of the smallest compression is increased, the action may approach more nearly that considered in the theories involving planes of least resistance. The tendency of the curves for the relation between lateral pressures and maximum unit loads (see Fig. 22) to have a slight curvature at low oil pressures, and to approach a straight line at higher pressures, agrees with this conception.

The average load—deformation curves of Fig. 23 for the specimens of Series 3A show that large deformations occur at maximum load with little change in the stress. This agrees with the conception of a disorganized material in which the main motion takes place in one direction only, resistance to the motion being a fairly constant cohesive resistance and a friction. Due to the development of splitting action and probably due to the large deformations which are likely to cause a reduction in both the cohesive and frictional resistance, the unit load carried by the specimens decreases slowly when the deformation is continued. This decrease is most pronounced for the specimens with the highest cohesive strength and tested at the smallest oil pressures, indicating that the reduction in unit load during large deformations is chiefly due to a reduction in cohesive strength, which is most apparent where the cohesive strength forms the largest part of the strength of the material.

VII. CONCLUSION

43. *Summary.*—A summary of the principal results of the foregoing analysis and tests is presented herewith. Many of the statements made are of a tentative nature, for, while the test results seem in many respects to warrant definite conclusions in view of their uniformity, in other respects an element of personal judgment enters into the interpretation of the test observations, particularly as regards the manner of failure. This summary is intended to give a general idea of the action of the material, supported by a fairly sound basis of experimental and analytical results, but with no claim of explaining directly all phases of the action of such a complex material as concrete.

(1) Tests of mortar and concrete covering the usual range of mixtures showed that in general the strength of the material in biaxial compression was as great as in simple compression, and that in many cases it was greater. The circumstances of the tests lead to the belief that, under more nearly identical test conditions in the two cases, the strength in biaxial compression would be found to be relatively greater than it was in these tests.

(2) The strength of the concrete in triaxial compression was found to increase greatly with the magnitude of the smallest principal stress. The tests failed to give conclusive evidence regarding the influence of the intermediate principal stress upon the strength, but indicated that this influence was not great.

(3) The tests of concrete of lean, medium, and rich mixtures in triaxial compression showed that the rate of increase in the strength with increase in the smallest principal stress was largely independent of the proportions of the concrete mixture.

(4) The triaxial compression tests showed that the presence of lateral pressures added to the strength of the specimen an amount approximately 4.1 times the magnitude of the lateral pressure. Thus the magnitude of the maximum principal stress developed was roughly equal to the strength of the concrete in simple compression plus 4.1 times the lateral pressure.

(5) The high stresses resisted by the concrete in triaxial compression (the maximum principal stress reaching 24 600 lb. per sq. in. in one instance) were always accompanied by very large deformations. The axial deformation at maximum load ranged from 0.5 per cent to more than 7.0 per cent of the length of the specimen, depending on the magnitude of the stress and the quality of the concrete.

(6) Much of the deformation under triaxial loading was due to an inelastic reduction in volume, or a compacting, of the concrete under stress. The amount of compacting varied considerably with the richness of the mixture.

(7) The tests of concrete in simple compression showed characteristic differences in behavior throughout three distinct stages of loading. In the first stage, the action was nearly elastic; in the second stage, an appreciable part of the deformation was inelastic and the action was marked by an increase in the rate of deformation and in the ratio of lateral to longitudinal deformation; in the third stage, which began at loads 75 to 85 per cent of the maximum load, a general breakdown of the internal continuity of the material developed. In this stage there was a very great increase in the lateral deformation, which finally produced an increase in volume under continued loading, indicating by this lateral bulging that a splitting failure was taking place throughout the material on surfaces parallel to the direction of the applied compressive stress.

(8) The tests in biaxial and triaxial compression also indicated that a process of splitting similar to that found in simple compression was prominent in the failure of the material under these combined stresses.

(9) The maximum tensile strain theory of failure of materials gives criteria of failure that are far from agreement with

the results of the tests, whether the theory is applied to the nominal deformations computed on the assumption of an elastic material, or to actual deformations measured at the beginning of failure under different combinations of stress.

(10) Many of the numerical results of the tests approach an agreement with the internal friction theory of sliding failure; however, the great increase in lateral deformations mentioned under (7) cannot be reconciled with a conception of failure as taking place through a sliding on plane surfaces continuous throughout the material. Hence it seems very doubtful whether Mohr's theory or any theory based on the assumption of a sliding on continuous planes of least resistance in a homogeneous material can give a correct representation of the failure of concrete in compression.

(11) The conception of failure advanced by Brandtzaeg and described in Sections 6 and 42 and in the appendix seems to give a reasonable picture of the process of failure of a material such as concrete. The many points of agreement found between analysis and tests lend very substantial support to the general hypothesis, and it appears that, as a means of explaining and correlating the many isolated and apparently unimportant phenomena of compression tests of concrete, this general conception of failure is particularly useful.

APPENDIX

ANALYSIS OF STRESSES IN A MATERIAL COMPOSED OF NON-ISOTROPIC ELEMENTS

BY

ANTON BRANDTZAEG

1. *General Relations.*—Reference has been made to a new conception of failure advanced by the author and qualitative discussions of its application are given in Sections 6 and 42. The following analysis of stresses and deformations existing in the assumed material at various stages of loading is essentially an abstract of the mathematical treatment presented in the original paper on the subject.*

*A. Brandtzaeg, "Failure of a Material Composed of Non-Isotropic Elements." Det Kgl. Norske Videnskabers Selskabs Skrifter. 1927. Nr. 2.

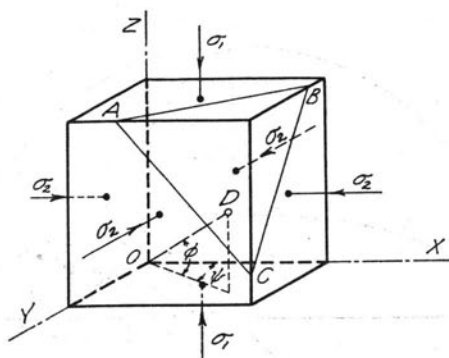


FIG. 33. LOCATION OF PLANES OF WEAKNESS OF AN ELEMENT OF VOLUME

As noted in Section 6, the assumed material is made up of small elements, each having planes of weakness in a certain direction. Each element is assumed to be initially perfectly elastic, having a modulus of elasticity E and Poisson's ratio μ , but after a certain limiting shearing stress has been reached in the direction of the planes of weakness of the element, a plastic sliding deformation takes place. Expressions will be given for the average normal stress f , for the stress in individual elements or intensity of stress σ , and for the deformation ϵ . The analysis applies to a cylindrical specimen subjected to an axial principal stress f_1 and to lateral principal stresses f_2 . Two cases of loading are considered which correspond to those of Series 3A and 3B and will be referred to as Case A, in which f_1 is greater than f_2 ; and Case B, in which f_1 is less than f_2 . It may be noted that the subscripts 1 and 2 used with stresses or deformations correspond to the directions of f_1 and f_2 ; further subscripts will also be used to distinguish between elastic and plastic action, the subscript γ referring to elastic action and the subscript β referring to plastic action. Compressive stresses and deformations will be considered as positive.

The cube of Fig. 33 may be considered as representing an element of volume of the material, for which the planes of weakness against plastic sliding are parallel to the plane ABC . The direction of this plane may be defined by the direction of OD (normal to the plane) expressed in terms of the angles ϕ and ψ . A convenient representation of all the elements in a unit volume, which contain all possible directions of weakness may be seen by imagining normals of unit length drawn in all possible directions from the origin of coördinates. All the end points of these normals fall on the surface of a sphere, and

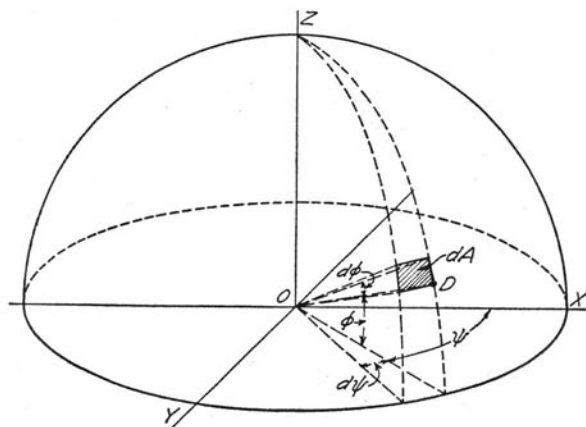


FIG. 34. REPRESENTATION OF ALL PLANES OF WEAKNESS IN A UNIT OF VOLUME

may thus be considered as representing the elements themselves and their directions of weakness. If the elements are cubes having a length of side d , the number of elements in a unit of volume is $N = \frac{1}{d^3}$. If this number of elements N is now represented by the surface of the hemisphere of Fig. 34, the number in a small portion of the surface dA is

$$dN = \frac{N}{2\pi} dA = \frac{1}{2\pi d^3} \cos \phi \, d\phi \, d\psi \quad (1)$$

Equation (1) expresses the manner in which the number of elements varies with the angles ϕ and ψ .

As stated in Section 6, the internal friction theory is assumed to apply to the plastic sliding deformation of individual elements. This may be expressed by the equation

$$\tau_{\text{lim}} = \tau_o + f\sigma \quad (2)$$

wherein τ_{lim} is the limiting value of shearing stress, τ_o is the shearing strength of the material, f is the coefficient of internal friction and σ is the normal stress. In determining the stresses in an element of volume, reference may be made to Fig. 1, Section 4, wherein the following expressions are given for the normal and shearing stresses, respectively, acting on a plane inclined at an angle ϕ with the vertical axis of the element:

$$\sigma = \sigma_1 \sin^2 \phi + \sigma_2 \cos^2 \phi$$

$$\tau = \frac{\sigma_1 - \sigma_2}{2} \sin 2\phi \quad (3)$$

By introducing the term $\sigma_v = \frac{1}{3}(\sigma_1 + 2\sigma_2)$ the expression for σ becomes

$$\sigma = \sigma_v + \frac{\sigma_1 - \sigma_2}{2} \left(\frac{1}{3} - \cos 2\phi \right) \quad (4)$$

Now by combining Equations (2), (3), and (4) and introducing an angle ϕ_o such that $f = \cot 2\phi_o = \tan \theta$, whence $\phi_o = \frac{\pi}{4} - \frac{\theta}{2}$, an expression is obtained for the difference in the principal stresses,

$$\left(\frac{\sigma_1 - \sigma_2}{2} \right)_{\text{lim}} = \frac{K + F\sigma_v}{\cos 2(\phi - \phi_o) - \frac{1}{3}F} \quad (5)$$

wherein K and F are functions of the constants τ_o and ϕ_o .

$$K = \tau_o \sin 2\phi_o; F = \cos 2\phi_o.$$

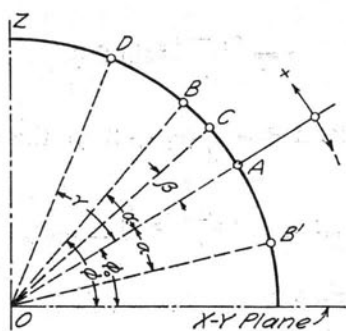
Equation (5) expresses a limiting value that the "stress difference" $\sigma_1 - \sigma_2$ cannot exceed; when this value is reached the element starts to deform plastically. Hence, if an element is deforming plastically, it is evident that the stress difference is at the limiting value. Equation (5) therefore expresses the necessary condition for plastic equilibrium.

With the application of load, the stress difference eventually reaches the limiting value for some elements, and plastic sliding deformation begins. Consideration of Equations (3) and (5) shows that the elements in which plastic action first develops are:

In Case A, $\sigma_1 > \sigma_2$, those for which $\phi = \phi_o$

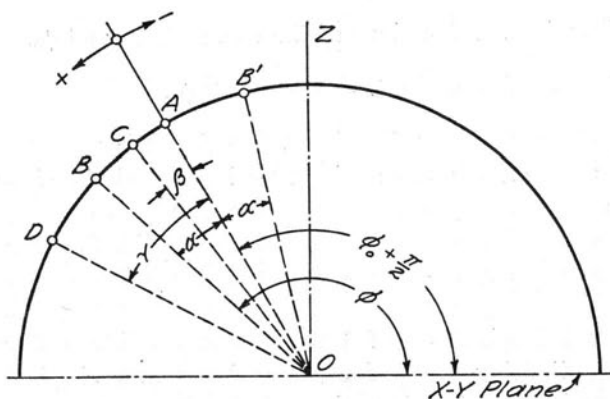
In Case B, $\sigma_1 < \sigma_2$, those for which $\phi = \frac{\pi}{2} + \phi_o$

As the application of load continues, plastic deformation spreads to those elements whose direction of weakness is near that for which plastic action first occurred. The spreading of plastic action may be studied by reference to Figs. 35 and 36, representing meridian circles on the hemisphere of Fig. 34. It is evident that the angle ψ has no influence upon the development of plasticity. Figure 35 refers to the loading of Case A. It is evident that elements in which plastic action



B to B', plastic elements
X-Y plane to B' } elastic elements
B to Z-axis }

FIG. 35. DIAGRAM SHOWING EXTENT OF PLASTIC AND ELASTIC ACTION, CASE A



B to B', plastic elements
Z-axis to B' } elastic elements
B to X-Y plane }

FIG. 36. DIAGRAM SHOWING EXTENT OF PLASTIC AND ELASTIC ACTION, CASE B

first starts are represented by the normal OA , for which $\phi = \phi_0$. The next elements to follow are represented by points up and down the meridian circle from A . Points B and B' , equidistant from A , represent such elements which are just passing into the plastic stage. The value of $\phi - \phi_0$ for these elements will be denoted by α , and the magnitude of α thus becomes a measure of the extent to which plasticity

has developed. Any point C between B and B' represents an element that is deforming plastically; its direction of weakness with reference to ϕ_0 is denoted by β . Any element outside the range $B - B'$ is elastic; its angle of weakness with reference to ϕ_0 is γ . It will be noted that β and γ are variable angles, while α has a certain value for any given degree of plastic development. The reasoning for Fig. 36, which applies to the loading of Case B, is exactly similar to the foregoing.

For plastic elements the value of the stress difference must always be given by Equation (5). Hence replacing the term $\cos 2(\phi - \phi_0)$ in Equation (5) by $\cos 2\beta$ for Case A, and by $-\cos 2\beta$ for Case B, gives

$$\text{For Case A: } \sigma_{\beta_1} - \sigma_{\beta_2} = 2 \frac{K + F\sigma_v}{\cos 2\beta - \frac{1}{3}F}$$

$$\text{For Case B: } \sigma_{\beta_1} - \sigma_{\beta_2} = -2 \frac{K + F\sigma_v}{\cos 2\beta + \frac{1}{3}F}$$

Since the deformation in all elements is the same, the stresses in elastic elements must be just equal to that in the element that is passing into the plastic stage, and hence is governed by equation (5). Equations for the stress differences in elastic elements will be similar to the two above, but with the angle β replaced by α . Now, from these equations for the stress difference and knowing that

$$\sigma_v = \frac{1}{3}(\sigma_1 + 2\sigma_2)$$

expressions for the principal stresses in the elements may be written in terms of the "volume stress," σ_v , and the angles α and β , as follows:

Case A:

In any elastic element:

$$\sigma_{\gamma_1} = \sigma_v + \frac{2}{3}(\sigma_{\gamma_1} - \sigma_{\gamma_2}) = \sigma_v + 4 \frac{K + F\sigma_v}{3 \cos 2\alpha - F} \quad (6a)$$

$$\sigma_{\gamma_2} = \sigma_v - \frac{1}{3}(\sigma_{\gamma_1} - \sigma_{\gamma_2}) = \sigma_v - 2 \frac{K + F\sigma_v}{3 \cos 2\alpha - F} \quad (6b)$$

In any plastic element:

$$\sigma_{\beta_1} = \sigma_v + 4 \frac{K + F\sigma_v}{3 \cos 2\beta - F} \quad (7a)$$

$$\sigma_{\beta_2} = \sigma_v - 2 \frac{K + F\sigma_v}{3 \cos 2\beta - F} \quad (7b)$$

Case B:

Elastic elements:

$$\sigma_{\gamma_1} = \sigma_v - 4 \frac{K + F\sigma_v}{3 \cos 2\alpha + F} \quad (8a)$$

$$\sigma_{\gamma_2} = \sigma_v + 2 \frac{K + F\sigma_v}{3 \cos 2\alpha + F} \quad (8b)$$

Plastic elements:

$$\sigma_{\beta_1} = \sigma_v - 4 \frac{K + F\sigma_v}{3 \cos 2\beta + F} \quad (9a)$$

$$\sigma_{\beta_2} = \sigma_v + 2 \frac{K + F\sigma_v}{3 \cos 2\beta + F} \quad (9b)$$

2. *Average Principal Stresses.*—From Equations (6a) to (9b) for the intensity of stress in all elements the average stress in a unit volume may be determined. The average axial unit stress f_1 , for example, must be equal to the sum of all unbalanced axial forces exerted by the elements in the unit volume. There are N elements exerting forces on each other in a unit volume. Since N has been taken as a large number, statistical averages may be applied; hence, $N(1-d)$ of the elementary forces must balance each other inside the unit volume, leaving the number of forces, N_f , to be balanced by f_1 equal to Nd . The same number of forces must be balanced by f_2 . The average principal stresses for the unit volume are now obtained by the summation of all unbalanced forces acting on elements. Using the relation $N_f = Nd$ in Equation (1) gives

$$dN_f = \frac{1}{2\pi d^2} \cos \varphi d\varphi d\psi \quad (10)$$

(a) *Case A.* Replacing ϕ in Equation (10) by $\gamma + \phi_o$ for elastic elements and $\beta - \phi_o$ for plastic elements, it is found that the elementary forces in dN_f elements are:

In an axial direction, exerted by elastic elements:

$$df_{\gamma_1} = \frac{1}{2\pi} \sigma_{\gamma_1} \cos(\gamma + \varphi_o) d\gamma d\psi$$

and exerted by plastic elements:

$$df_{\beta_1} = \frac{1}{2\pi} \sigma_{\beta_1} \cos(\beta + \varphi_o) d\beta d\psi$$

In a lateral direction, exerted by elastic elements:

$$df_{\gamma_2} = \frac{1}{2\pi} \sigma_{\gamma_2} \cos(\gamma + \varphi_o) d\gamma d\psi$$

and exerted by plastic elements:

$$df_{\beta_2} = \frac{1}{2\pi} \sigma_{\beta_2} \cos(\beta + \varphi_o) d\beta d\psi$$

The average principal stresses may now be obtained by a double integration with respect to ψ and γ or β , integrating with respect to ψ between limits of 0 and 2π and with respect to γ and β from $\phi = 0$ to $\phi = \frac{\pi}{2}$. The following expressions for the average principal stresses for Case A are obtained.

$$f_1 = \sigma_v + 4R_a(K + F\sigma_v) \quad (11a)$$

$$f_2 = \sigma_v - 2R_a(K + F\sigma_v) \quad (11b)$$

$$f_v = \frac{1}{3}(f_1 + 2f_2) = \sigma_v \quad (11c)$$

R_a is the following function of α :

$$R_a = \frac{1 - 2 \cos \varphi_o \sin \alpha}{3 \cos 2\alpha - F} + \frac{\cos \varphi_o}{6 \sqrt{\frac{1}{2} - \frac{1}{6}F}} \log_e \frac{\sqrt{\frac{1}{2} - \frac{1}{6}F + \sin \alpha}}{\sqrt{\frac{1}{2} - \frac{1}{6}F - \sin \alpha}} \quad (12)$$

Combining Equations (11a) and (11b) to eliminate σ_v gives:

$$f_1 = f_2 \frac{1 + 4FR_a}{1 - 2FR_a} + K \frac{6R_a}{1 - 2FR_a} \quad (13a)$$

$$f_2 = f_1 \frac{1 - 2FR_a}{1 + 4FR_a} - K \frac{6R_a}{1 + 4FR_a} \quad (13b)$$

$$f_v = \sigma_v = \frac{f_1 - 4KR_a}{1 + 4FR_a} = \frac{f_2 + 2KR_a}{1 - 2FR_a} \quad (13c)$$

(b) *Case B.* By a similar procedure we obtain the relations which apply in the case when $f_1 < f_2$.

$$f_1 = f_2 \frac{1 - 4FQ_a}{1 + 2FQ_a} - K \frac{6Q_a}{1 + 2FQ_a} \quad (14a)$$

$$f_2 = f_1 \frac{1 + 2FQ_a}{1 - 4FQ_a} + K \frac{6Q_a}{1 - 4FQ_a} \quad (14b)$$

$$f_v = \sigma_v = \frac{f_2 - 2KQ_a}{1 + 2FQ_a} = \frac{f_1 + 4KQ_a}{1 - 4FQ_a} \quad (14c)$$

Here Q_a is the following function of α :

$$Q_a = \frac{1 - 2\sin\varphi_o \sin\alpha}{3\cos 2\alpha + F} + \frac{\sin\varphi_o}{6\sqrt{\frac{1}{2} + \frac{1}{6}F}} \log_e \frac{\sqrt{\frac{1}{2} + \frac{1}{6}F} + \sin\alpha}{\sqrt{\frac{1}{2} + \frac{1}{6}F} - \sin\alpha} \quad (15)$$

3. *Final Expressions for Stresses in Individual Elements.*—By substituting values of σ_v from Equations (13c) and (14c) in Equations (6a) to (9b) the stresses in individual elements may now be expressed in terms of the smaller of the average principal stresses, and the angle α , which defines the general stress situation in the material.* The stresses in plastic elements are, of course, dependent upon their individual directions of weakness, defined by the angle β .

(a) *Case A:*

$$\sigma_{\gamma_1} = f_2 \frac{3(\cos 2\alpha + F)}{(1 - 2FR_a)(3\cos 2\alpha - F)} + K \frac{4 + 2R_a(3\cos 2\alpha - F)}{(1 - 2FR_a)(3\cos 2\alpha - F)} \quad (16a)$$

$$\sigma_{\gamma_2} = f_2 \frac{3(\cos 2\alpha - F)}{(1 - 2FR_a)(3\cos 2\alpha - F)} - K \frac{2 - 2R_a(3\cos 2\alpha - F)}{(1 - 2FR_a)(3\cos 2\alpha - F)} \quad (16b)$$

$$\sigma_{\beta_1} = f_2 \frac{3(\cos 2\beta + F)}{(1 - 2FR_a)(3\cos 2\beta - F)} + K \frac{4 + 2R_a(3\cos 2\beta - F)}{(1 - 2FR_a)(3\cos 2\beta - F)} \quad (16c)$$

$$\sigma_{\beta_2} = f_2 \frac{3(\cos 2\beta - F)}{(1 - 2FR_a)(3\cos 2\beta - F)} - K \frac{2 - 2R_a(3\cos 2\beta - F)}{(1 - 2FR_a)(3\cos 2\beta - F)} \quad (16d)$$

*A simpler form of the final expressions for the stresses in individual elements has recently been derived by Dr. Edgar B. Schieldrop and published in two notes, "Brandtzaeg's Theory Mathematically Simplified" and "Simple Graphical Solutions of Stress Problems Afforded by Brandtzaeg's Theory of Failure," Det Kongelige Norske Videnskabers Selskab, Forhandling Bd. 1, Nr. 32 and 33, Trondhjem, Norway, 1928. These papers contain no essential change in the theory involved, but do introduce a graphical construction which greatly simplifies the mathematical treatment.

(b) *Case B:*

$$\sigma_{\gamma_1} = f_1 \frac{3(\cos 2\alpha - F)}{(1 - 4FQ_a)(3 \cos 2\alpha + F)} - K \frac{4 - 4Q_a(3 \cos 2\alpha + F)}{(1 - 4FQ_a)(3 \cos 2\alpha + F)} \quad (17a)$$

$$\sigma_{\gamma_2} = f_1 \frac{3(\cos 2\alpha + F)}{(1 - 4FQ_a)(3 \cos 2\alpha + F)} + K \frac{2 + 4Q_a(3 \cos 2\alpha + F)}{(1 - 4FQ_a)(3 \cos 2\alpha + F)} \quad (17b)$$

$$\sigma_{\beta_1} = f_1 \frac{3(\cos 2\beta - F)}{(1 - 4FQ_a)(3 \cos 2\beta + F)} - K \frac{4 - 4Q_a(3 \cos 2\beta + F)}{(1 - 4FQ_a)(3 \cos 2\beta + F)} \quad (17c)$$

$$\sigma_{\beta_2} = f_1 \frac{3(\cos 2\beta + F)}{(1 - 4FQ_a)(3 \cos 2\beta + F)} + K \frac{2 + 4Q_a(3 \cos 2\beta + F)}{(1 - 4FQ_a)(3 \cos 2\beta + F)} \quad (17d)$$

4. *Final Expressions for Deformations.*—As long as the material acts elastically, the axial unit deformation ϵ_1 of an element and the lateral unit deformation ϵ_2 are given by the well-known equations:

$$\epsilon_1 = \frac{1}{E}(\sigma_1 - 2\mu\sigma_2)$$

$$\epsilon_2 = \frac{1}{E}(\sigma_2 - \mu(\sigma_1 + \sigma_2))$$

If the stress, $\sigma_v = \frac{1}{3}(\sigma_1 + 2\sigma_2)$ is introduced, the equations become:

$$\epsilon_1 = \frac{1 - 2\mu}{E}\sigma_v + \frac{2}{3}\frac{1 + \mu}{E}(\sigma_1 - \sigma_2) \quad (18a)$$

$$\epsilon_2 = \frac{1 - 2\mu}{E}\sigma_v - \frac{1}{3}\frac{1 + \mu}{E}(\sigma_1 - \sigma_2) \quad (18b)$$

Now although part of the assumed material is acting plastically, since it has been assumed that all elements deform a like amount, it follows that the deformation of the material as a whole is equal to the deformation of the elastic elements. Hence, substituting the values of the stress difference found by the application of Equations (5) and the values of the volume stress σ_v from Equations (13c) and (14c), the following expressions for the axial and lateral deformations are found:

(a) *Case A:*

$$\epsilon_1 = \frac{1}{E} \left[3f_1 \frac{(1-2\mu) \cos 2\alpha + (1+2\mu) F}{(3 \cos 2\alpha - F) (1 - 2F R_a)} + K \frac{4(1+\mu) + 2(1-2\mu) R_a (3 \cos 2\alpha - F)}{(3 \cos 2\alpha - F) (1 - 2F R_a)} \right] \quad (19a)$$

$$\epsilon_2 = \frac{1}{E} \left[3f_2 \frac{(1-2\mu) \cos 2\alpha - F}{(3 \cos 2\alpha - F) (1 - 2F R_a)} - K \frac{2(1+\mu) - 2(1-2\mu) R_a (3 \cos 2\alpha - F)}{(3 \cos 2\alpha - F) (1 - 2F R_a)} \right] \quad (19b)$$

(b) *Case B:*

$$\epsilon_1 = \frac{1}{E} \left[3f_1 \frac{(1-2\mu) \cos 2\alpha - (1+2\mu) F}{(3 \cos 2\alpha + F) (1 - 4F Q_a)} - K \frac{4(1+\mu) - 4(1-2\mu) Q_a (3 \cos 2\alpha + F)}{(3 \cos 2\alpha + F) (1 - 4F Q_a)} \right] \quad (20a)$$

$$\epsilon_2 = \frac{1}{E} \left[3f_2 \frac{(1-2\mu) \cos 2\alpha + F}{(3 \cos 2\alpha + F) (1 - 4F Q_a)} + K \frac{2(1+\mu) + 4(1-2\mu) Q_a (3 \cos 2\alpha + F)}{(3 \cos 2\alpha + F) (1 - 4F Q_a)} \right] \quad (20b)$$

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